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Generalization of Hilbert's Integral Inequality

Generalización de la desigualdad integral de Hilbert

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Abstract. In this paper we make some further generalization of well known Hilbert's inequality and its equivalent form in two dimensional case. Many other results of this type in recent years follows as a special case of our results.

Key words and phrases. Hardy-Hilbert's inequality, Best constant, Hölder's inequality, Minkowski's integral inequality.

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Resumen. En este artículo se hace una generalización de la conocida desigualdad de Hilbert y su forma equivalente en el caso de dos dimensiones. Otros resultados de este tipo de años recientes, se siguen como un caso especial de los resultados aquí presentados.

Palabras y frases clave. Desigualdad de Hardy-Hilbert, mejor constante, desigualdad de Hölder, desigualdad integral de Minkowski.

1. Introduction

Hilbert's classic inequality

$$\sum_m \sum_n \frac{a_n b_n}{m+n} \le \pi \Big(\sum_n a_n^2 \Big)^{1/2} \Big(\sum_m b_n^2 \Big)^{1/2}$$

and variants and generalizations of it found application in theory of number, specially in connection with the large sieve (method of analytic number theory). During the past century Hilbert's inequality was generalized in many different directions. Similar inequalities, in operator form, appear in harmonic analysis where one investigate properties of such operators. Next, we recall the Hilbert inequality and the Hardy–Hilbert inequality. If f, g are real functions such that $0 < \int_0^\infty f^2(x)\,dx < \infty$, and $\int_0^\infty g^2(x)\,dx < \infty$, then we have (see [4])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y}\,dx\,dy < \pi\|f\|_2\|g\|_2, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) is the well known Hilbert's inequality. Inequality (1) had been generalized by Hardy–Riesz (see [3]) in 1920 as:

If f, g are non-negative real functions such that $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $\int_0^\infty g^q(x)\,dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_p \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. When $p = q = 2$, (2) reduces to (1). Under the same conditions of (1) and (2) inequalities (1) and (2) are equivalent to the following two inequalities

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y}\,dx\right]^2 dy < \pi^2 \int_0^\infty f^2(x)\,dx, \tag{3}$$

and

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y}\,dx\right]^p dy < \left[\frac{\pi}{\sin(\pi/p)}\right]^p \int_0^\infty f^p(x)\,dx, \tag{4}$$

where the constant factors π^2 and $\left[\frac{\pi}{\sin(\pi/p)}\right]^p$ are the best possible (see [4]).

In recent years a lot of results with generalization of the Hardy–Hilbert integral inequality were obtained (see [1, 2, 6, 8, 9]). Let us mention some of them which take our attention. Li Wu and He [7] obtained the following inequality.

Theorem 1. *If f, g are non-negative real functions such that*

$$0 < \int_0^\infty f^2(x)\,dx < \infty \qquad \text{and} \qquad \int_0^\infty g^2(x)\,dx < \infty,$$

then we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}}\,dx\,dy < C\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(x)\,dx\right)^{\frac{1}{2}}, \tag{5}$$

where the constant factor $C = 1.7408\cdots$ is the best possible.

B. He, Y. Li and Y. Qian (see [5]) obtained the following inequality:

Theorem 2. *If f, g are real functions such that $0 < \int_0^\infty f^2(x)\,dx < \infty$, and $0 < \int_0^\infty g^2(x)\,dx < \infty$. Then we have*

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x+y} f(x)g(y) < A\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(x)\,dx\right)^{\frac{1}{2}}, \quad (6)$$

where $A = 7.3277\cdots$ is the best possible.

Recently B. He, Y. Li and Y. Qian [5] obtained the following inequality.

Theorem 3. *If f, g are real functions such that $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(x)\,dx < \infty$. Then we have*

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x+y+\min\{x,y\}} f(x)g(y) < A\left(\int_0^\infty f^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_0^\infty g^2(x)\,dx\right)^{\frac{1}{2}}, \quad (7)$$

where $A = 6.88947\cdots$ is the best possible.

2. Main Results

To proof our main result, we use the standard methods of [2] section 6.3.

Theorem 4. *Suppose K is Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that*

a) $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$, for all $\lambda > 0$ (homogeneous of degree -1).

b) $\int_0^\infty |K(x,1)|x^{\frac{1}{q}-1}\,dx = A < \infty$, where $1 \le p \le \infty$ and q is the conjugate exponent to p. If $f \in L_p(0,\infty)$ and $g \in L_q(0,\infty)$, then

$$\left|\int_0^\infty \int_0^\infty K(x,y)f(y)g(x)\,dx\,dy\right| < A\|f\|_p\|g\|_q. \quad (8)$$

Proof. For fixed y, setting $x = zy$, and putting into the integral below, we have.

$$\begin{aligned}\int_0^\infty \int_0^\infty |K(x,y)f(x)g(y)|\,dx\,dy &= \int_0^\infty \int_0^\infty |K(zy,y)f(zy)g(y)|\,dz\,dy\\ &= \int_0^\infty \int_0^\infty |K(z,1)f(yz)g(y)|\,dz\,dy\end{aligned}$$

therefore, Fubini's theorem, Hölder's inequality and a suitable change of variable gives

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty |K(z,1)f(x)g(y)|\,dx\,dy\\
&\qquad = \int_0^\infty \int_0^\infty |K(z,1)f(yz)g(y)|\,dz\,dy\\
&\qquad = \int_0^\infty \int_0^\infty \left|K(z,1)f(u)g\Big(\frac{u}{z}\Big)z^{-1}\right|\,dz\,du\\
&\qquad \le \int_0^\infty |K(z,1)|z^{-1}\Big(\int_0^\infty f(u)\,du\Big)^{\frac{1}{p}}\Big(\int_0^\infty \Big|g\Big(\frac{u}{z}\Big)\Big|^q\,du\Big)^{\frac{1}{q}}\,dz\\
&\qquad = \Big(\int_0^\infty |K(z,1)|z^{1/q-1}\,dz\Big)\|f\|_p\|g\|_q,
\end{aligned}
$$

hence

$$
\left|\int_0^\infty \int_0^\infty |K(x,y)f(y)g(x)|\,dx\,dy\right| \le A\|f\|_p\|g\|_q. \tag{9}
$$

Note that equality in (9) can only occur if f or g is null or both are effectively proportional.

The first possibility would contradict one of the hypotheses; the second possibility implies that for almost all x and all y there exist constants c and d they are not all zero, without loss of generality, suppose $c \neq 0$, and

$$
cK(x,y)[f(x)]^p x^{\frac{1}{p}-1} = dK(x,y)[g(y)]^q y^{\frac{1}{q}-1},
$$

i.e. in $(0,\infty)\times(0,\infty)$. Then we have obtain

$$
c[f(x)]^p x^{\frac{1}{p}-1} = d[g(y)]^q y^{\frac{1}{q}-1} = \text{constant},
$$

i.e. in $(0,\infty)\times(0,\infty)$. Thus,

$$
\int_0^\infty [f(x)]^p\,dx = \frac{\text{const}}{c}\int_0^\infty \frac{dx}{x^{\frac{1}{p}-1}},
$$

which contradict the fact that $f \in L_p(0,\infty)$. Hence (9) takes the form of strict inequality, so we have (8). This completes the proof of Theorem 4. ☑

Theorem 5. *Suppose K is a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that*

a) $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ for all $\lambda > 0$, and

b) $\int_0^\infty |K(x,1)|x^{\frac{1}{p}}\,dx = C < \infty$, where $1 \le p \le \infty$. If $f \in L_p$, then

$$
\int_0^\infty \left|\int_0^\infty K(x,y)f(x)\,dx\right|^p dy < A^p \int_0^\infty |f(x)|^p\,dx. \tag{10}
$$

Proof. The proof goes line by line the same as the proof of Theorem (4), but in place of Hölder's inequality we use the Minkowski inequality for integral (see [2]). ☑

Theorem 6. *Inequality* (8) *is equivalent to* (10).

Proof. Suppose that inequality (8) holds, then we let $g(y) = \int_0^\infty K(x,y)f(x)\,dx$. By Theorem (5) we see that $g^{p-1} \in L_q$. Then

$$\begin{aligned}
\int_0^\infty |g(y)|^p\,dy &= \int_0^\infty |g(y)||g(y)|^{p-1}\,dy \\
&= \int_0^\infty \left|\int_0^\infty K(x,y)f(x)\,dx\right||g(y)|^{p-1}\,dy \\
&\le \int_0^\infty \int_0^\infty |K(x,y)f(x)||g(y)|^{p-1}\,dx\,dy \\
&< A\Big(\int_0^\infty |f(x)|^p\,dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty |g(y)|^p\,dy\Big)^{\frac{1}{q}},
\end{aligned}$$

thus,

$$\int_0^\infty |g(y)|^p\,dy < A^p \int_0^\infty |f(x)|^p\,dx.$$

Conversely if inequality(10) holds, then

$$\begin{aligned}
\int_0^\infty \int_0^\infty |K(x,y)f(x)g(y)|\,dx\,dy &= \int_0^\infty (|K(x,y)f(x)|\,dx)|g(y)|\,dy \\
\le \Big(\int_0^\infty \Big(\int_0^\infty |K(x,y)f(x)|\,dx\Big)^p dy\Big)^{\frac{1}{p}} &\Big(\int_0^\infty |g(y)|^q\,dy\Big)^{\frac{1}{q}} \\
&< A\|f\|_p\|g\|_q,
\end{aligned}$$

as we claimed. ☑

***Remark* 1.**

a) If $K(x,y) = \frac{1}{x+y}$ and $p = q = 2$ then (1) follows as special case of (8).

b) Also, for $K(x,y) = \frac{1}{x+y}$ and for any $1 \le p < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, (2) follows as special case of (8).

c) If $K(x,y) = \frac{1}{x+y+\max\{x,y\}}$ and $p = q = 2$ from (8) we have (5) as special case, the same happen if $K(x,y) = \frac{|\ln x - \ln y|}{x+y}$.

d) Finally, if $K(x,y) = \frac{|\ln x - \ln y|}{x+y+\min\{x,y\}}$ and $p = q = 2$ from (8) we have (7).

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