A NOTE ON GENERALIZED MOBIUS $\mu$-FUNCTIONS

by

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In [1] the concept of a conjugate pair of sets of positive integers is introduced. Briefly, if $Z$ denotes the set of positive integers and $P$ and $Q$ denote non-empty subsets of $Z$ such that: if $n_1 \in Z$, $n_2 \in Z$, $(n_1, n_2) = 1$, then

(1) $n = n_1 n_2 \in P$ (resp. $Q$) $\iff n_1 \in P, n_2 \in P$ (resp. $Q$),

and, if in addition, for each integer $n \in Z$ there is a unique factorization of the form

(2) $n = ab$, $a \in P$, $b \in Q$,

we say that each of the sets $P$ and $Q$ is a direct factor set of $Z$, and that $(P, Q)$ is a conjugate pair. It is clear that $P \cap Q = \{1\}$. Among the generalized functions studied in [1], we find

(3) $\mu_P(n) = \sum_{d|n} \mu(n/d)$

$\mu_Q(n/d)$

de$\in P$

de$\in Q$

a generalization of MOBIUS $\mu$-function. The following results are also proved in [1]:

(i) $\mu_P$ is a multiplicative function.

(ii) $\sum_{d|n} \mu_P(n/d) = \rho(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$
Here we shall show that \( \mu_p \) is the unique arithmetical function satisfying (ii) above. Let \( \mu^* \) be such that

\[
(4) \quad \sum_{d|n} \mu^*(n/d) = \varphi(n) = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{if } n > 1 
\end{cases}
\]

If \( \mu^* \) is multiplicative, it suffices to prove that \( \mu_p(p^k) = \mu^*(p^k) \) for every prime \( p \) and every integer \( k > 0 \). So let \( \mu^* \) be a multiplicative function; it follows from (4) and (ii) that \( \mu_p(1) = \mu^*(1) \), thus \( \mu_p(p) = \mu^*(p) \) for every prime \( p \). We will now show by induction on \( k \) that \( \mu_p(p^k) = \mu^*(p^k) \). Suppose this relation holds for \( u > k > 0 \). From (ii) we obtain

\[
(5) \quad \mu_p(p^u) = -\sum_{1 < i < u} \mu_p(p^u/p^i),
\]

because \( 1 = p^0 \in \mathbb{Q} \). On the other hand, from (4) we obtain

\[
(6) \quad \mu^*(p^u) = -\sum_{1 < i < u} \mu^*(p^u/p^i),
\]

because \( 1 = p^0 \in \mathbb{Q} \). But by the induction hypothesis, \( \mu_p(p^{u-i}) = \mu^*(p^{u-i}) \) \( (i = 1, \ldots, u) \). Thus the right members in (5) and (6) are equal, so that \( \mu_p(p^u) = \mu^*(p^u) \).

In view of the above result, it suffices to show that any function \( \mu^* \) satisfying (4) is multiplicative, thus proving the following

**Theorem 1.** If \( \mu^* \) satisfies (4) for every \( n \in \mathbb{Z} \), then \( \mu^* = \mu_p \).
In order to prove this theorem we begin with some lemmata. ([2])

**Lemma 1.** Let $g$ be a multiplicative function. If $g(1) = 0$ $\implies$ $g(n) = 0$ for every $n \in \mathbb{Z}$. If $g(1) \neq 0$ $\implies$ $g(1) = 1$.

**Lemma 2.** Let $f$ be an arithmetical function. If $\sum_{d|n, d \in \mathbb{Q}} f(n/d) = 0$ for every $n \in \mathbb{Z}$, then $f(n) = 0$ for every $n \in \mathbb{Z}$.

Proof: As in [1, lemma 2], we proceed by induction on $n$, noting that $f(n)$ always appears in the involved sum, because $1 \in \mathbb{Q}$.

**Lemma 3.** Let $g$ be a multiplicative function. If $f$ is such that

$$g(n) = \sum_{d|n} f(n/d)$$

$d \in \mathbb{Q}$

then $f$ is multiplicative.

Proof: The proof is a convenient and trivial adaptation of that of lemma 3, [2]. For the sake of clearness we repeat it. If $g(1) = 0$, then $g(n) = 0$ for every $n \in \mathbb{Z}$, so by lemma 2, $f(n) = 0$ for every $n \in \mathbb{Z}$. If $g(1) = 1$ it is clear that that $f(1) = 1$. Let us consider the following proposition:

$$P_{m,n}^{\frac{1}{2}} : \langle (m,n)=1 \implies f(mn) = f(m)f(n) \rangle.$$

If $m$ or $n = 1$, the above proposition is true. Let us suppose that $P_{m,n}^{\frac{1}{2}}$ is false for some pair $(m,n)$, and let

$$m_0 = \min \{ m; \exists n \in \mathbb{Z} \text{ with } (m,n)=1 \text{ such that } P_{m,n}^{\frac{1}{2}} \text{ is false} \}$$
then there exists an \( n \) such that \( (m_0, n) = 1 \) and \( P\{m_0, n\} \) is false. Now let

\[
   n_0 = \min \{ n; (m_0, n) = 1 \text{ and } P\{m_0, n\} \text{ is false} \}.
\]

We have then:

i) \( 1 < m_0 < n_0 \) and \( (m_0, n_0) = 1 \).

ii) \( P\{m_0, n_0\} \) is false.

iii) \( P\{k, n\} \) is true for every \( n \) and each \( k \) such that \( 1 \leq k < m_0 \) and \( (k, n) = 1 \).

iv) \( P\{m_0, t\} \) is true for each \( t \) such that \( 1 \leq t \leq n_0 \) and \( (m_0, t) = 1 \).

If now we take \( g(m_0, n_0) \) we find

\[
   \sum_{t|m_0n_0} f(m_0n_0/t) = g(m_0, n_0) = g(m_0)g(n_0)
\]

\[
   = \left\{ \sum_{d|m_0} f(m_0/d) \right\} \times \left\{ \sum_{\delta|n_0} f(\delta/n_0) \right\}
\]

\[
   t \in \mathbb{Q}
\]

\[
   d \in \mathbb{Q}
\]

\[
   \delta \in \mathbb{Q}
\]

using the multiplicativity of \( g \) and (1); so that

\[
   \sum_{d|m_0, \delta|n_0} \left\{ f(m_0n_0/d\delta) - f(m_0/d)f(n_0/\delta) \right\} = 0
\]

But \( m_0/d \) and \( n_0/\delta \) are smaller than \( m_0 \) and \( n_0 \), resp., if \( d, \delta \neq 1 \); thus from the above relation and the hypothesis on \( (m_0, n_0) \), we conclude that

\[
   f(m_0n_0) - f(m_0)f(n_0) = 0,
\]

contradicting ii). So \( P\{m, n\} \) is always true.

We remark that no explicit calculation for \( \mu_P \) is
is needed in the above reasonings. Further, if we use the following theorem [1, theorem 3]:

**THEOREM 2.** If \( f \) and \( g \) are arithmetical functions, then

$$
(iii) \quad g(n) = \sum_{d|n} f(n/d) \iff f(n) = \sum_{d|n} g(d) \mu_p(n/d).
$$

the uniqueness of \( \mu_p \) is easily proved. Here we have proved Theorem 1 without help of this result; but we will prove more: \( \mu_p \) is the sole function that can perform the inversion in Theorem 2. For this we have to prove the following

**LEMMA 4.** If \( f(1) \neq 0 \) and \( \sum_{e|n} f(e) \rho^*(n/e) = f(n) \), then \( \rho^*(n) = \rho(n) \) for every \( n \in \mathbb{Z} \).

**Proof:** See [2, lemma 4].

Suppose now that \( \mu^* \) is such that

$$
(7) \quad g(n) = \sum_{d|n} f(n/d) \iff f(n) = \sum_{d|n} g(d) \mu^*(n/d).
$$

Then

$$
f(n) = \sum_{d|n} g(d) \mu^*(n/d) = \sum_{d|n} \mu^*(n/d) \cdot \sum_{e|n} f(e) = \sum_{e|n} \mu^*(\delta^*) = \sum_{e|n} \mu^*(\delta) = \sum_{\delta|e} \delta = n/e = \sum_{\delta|e} \mu^*(\delta).
$$

writing \( \rho^*(n) = \sum_{n=\delta d, d|e} \mu^*(\delta) \) we have \( f(n) = \sum_{e|n} f(e) \rho^*(n/e) \), so by lemma 4 and theorem 1,

$$
\rho(n) = \rho^*(n) = \sum \mu^*(\delta) = \mu^* = \mu_p.
$$

Thus we have proved the
Theorem 3. Let $f$ and $g$ be arithmetical functions such that $g(1) \neq 0$. If

$$g(n) = \sum_{d|n} f(n/d) \iff f(n) = \sum_{d|n} g(d) \mu^*(n/d)$$

then $\mu^* = \mu_P$.

References


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