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A NOTE ON GENERALIZED MOBIUS μ-FUNCTIONS

by

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In [1] the concept of a conjugate pair of sets of positive integers is **introduced**. Briefly, if Z denotes the set of positive integers and P and Q denote non-empty subsets of Z such that: if $n_1 \in Z$, $n_2 \in Z$, $(n_1, n_2) = 1$, then

(1) $n = n_1 n_2 \in P(\text{resp.} Q) \Longleftrightarrow n_1 \in P, n_2 \in P$ (resp. Q),

and, if in addition, for each integer $n \in Z$ there is a unique factorization of the form

(2) $n = ab$, $a \in P$, $b \in Q$,

we say that each of the sets P and Q is a <u>direct factor set of</u> Z, and that (P,Q) is a <u>conjugate pair</u>. It is clear that $P \cap Q = \{1\}$. Among the generalized functions studied in [1], we find

$$(3) \quad \mu_P(n) = \sum_{\substack{d \mid n \\ d \in P}} \mu(n/d)$$

a generalization of MOBIUS μ-function. The following results are also proved in [1]:

(i) μ_P <u>is a multiplicative function</u>.

(ii)
$$\sum_{\substack{d \mid n \\ d \in Q}} \mu_P(n/d) = \rho(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Here we shall show that μ_P is the unique arithmetical function satisfying (ii) above. Let μ^* be such that

$$(4) \qquad \sum_{\substack{d|n \\ d\in Q}} \mu^*(n/d) = \rho(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

If μ^* is multiplicative, it suffices to prove that $\mu_P(p^k) = \mu^*(p^k)$ for every prime p and every integer $k > 0$. So let μ^* be a multiplicative function; it follows from (4) and (ii) that $\mu_P(1) = \mu^*(1)$, thus $\mu_P(p) = \mu^*(p)$ for every prime p. We will now show by induction on k that $\mu_P(p^k) = \mu^*(p^k)$. Suppose this relation holds for $u > k \geq 0$. From (ii) we obtain

$$(5) \qquad \mu_P(p^u) = - \sum_{\substack{1\leq i\leq u \\ p^i\in Q}} \mu_P(p^u/p^i) \quad ,$$

because $1 = p^o \in Q$. On the other hand, from (4) we obtain

$$(6) \qquad \mu^*(p^u) = - \sum_{\substack{1\leq i\leq u \\ p^i\in Q}} \mu^*(p^u/p^i) \quad ,$$

because $1 = p^o \in Q$. But by the induction hypothesis, $\mu_P(p^{u-i}) = \mu^*(p^{u-i})$ ($i = 1,\ldots, u$). Thus the rigth members in (5) and (6) are equal, so that $\mu_P(p^u) = \mu^*(p^u)$

In view of the above result, it suffices to show that any function μ^* satisfying (4) is multiplicative, thus proving the following

THEOREM 1. *If* μ^* *satisfies* (4) *for every* $n \in Z$, *then* $\mu^* = \mu_P$.

In order to prove this theorem we begin with some lemmata. ([2])

LEMMA 1. Let g be a multiplicative function. If $g(1) = 0 \Rightarrow g(n) = 0$ for every $n \in Z$. If $g(1) \neq 0 \Rightarrow g(1) = 1$.

LEMMA 2. Let f be an arithmetical function. If $\Sigma_{d|n, d\in Q} f(n/d) = 0$ for every $n \in Z$, then $f(n) = 0$ for every $n \in Z$.

Proof: As in [1, lemma 2], we proceed by induction on n, noting that $f(n)$ always appears in the involved sum, because $1 \in Q$.

LEMMA 3. Let g be a multiplicative function. If f is such that

$$g(n) = \sum_{\substack{d|n \\ d\in Q}} f(n/d)$$

then f is multiplicative.

Proof: The proof is a convenient and trivial adaptation of that of lemma 3, [2]. For the sake of clearness we repeat it. If $g(1) = 0$, then $g(n) = 0$ for every $n \in Z$, so by lemma 2, $f(n) = 0$ for every $n \in Z$. If $g(1) = 1$ it is clear that that $f(1) = 1$. Let us consider the following proposition:

$$P\{m,n\} : \text{<<} \ (m,n)=1 \Rightarrow f(mn) = f(m)f(n) \ \text{>>}.$$

If m or $n = 1$, the above proposition is true. Let us suppose that $P\{m,n\}$ is false for some pair (m,n), and let

$$m_o = \min\{m;\ \exists\ n\in Z \text{ with } (m,n)=1 \text{ such that } P\{m,n\} \text{ is false}\}$$

then there exists an n such that $(m_o,n) = 1$ and $P\{m_o,n\}$ is false. Now let

$$n_o = \min\left\{n;\ (m_o,n)=1 \text{ and } P\{m_o,n\} \text{ is false}\right\}.$$

We have then:

i) $1 < m_o < n_o$ and $(m_o,n_o) = 1$.

ii) $P\{m_o,n_o\}$ is false.

iii) $P\{k,n\}$ is true for every n and each k such that $1 \leq k < m_o$ and $(k,n) = 1$.

iv) $P\{m_o,t\}$ is true for each t such that $1 \leq t \leq n_o$ and $(m_o,t) = 1$.

If now we take $g(m_on_o)$ we find

$$\sum_{\substack{t|m_on_o \\ t\in Q}} f(m_on_o/t) = g(m_on_o) = g(m_o)g(n_o)$$

$$= \left\{\sum_{\substack{d|m_o \\ d\in Q}} f(m_o/d)\right\} \times \left\{\sum_{\substack{\delta|n_o \\ \delta\in Q}} f(n_o/\delta)\right\}$$

using the multiplicativity of g and (1); so that

$$\sum_{\substack{d|m_o,\delta|n_o \\ d,\delta\in Q}} \left\{f(m_on_o/d\delta) - f(m_o/d)f(n_o/\delta)\right\} = 0$$

But m_o/d and n_o/δ are smaller than m_o and n_o, resp., if $d,\delta \neq 1$; thus from the above relation and the hypothesis on (m_o,n_o), we conclude that

$$f(m_on_o) - f(m_o)f(n_o) = 0,$$

contradicting ii). So $P\{m,n\}$ is always true.

We remark that no explicit calculation for μ_P is is needed in the above reasonings. Further, if we use the following theorem [1, theorem 3]:

THEOREM 2. *If f and g are arithmetical functions, then*

$$\text{(iii)} \quad g(n) = \sum_{\substack{d|n \\ d\in Q}} f(n/d) \Leftrightarrow f(n) = \sum_{d|n} g(d)\mu_P(n/d).,$$

the uniqueness of μ_P is easily proved. Here we have proved Theorem 1 without help of this result; but we will prove more: μ_P is the sole function that can perform the inversion in Theorem 2. For this we have to prove the following

LEMMA 4. *If* $f(1) \neq 0$ *and* $\Sigma_{e|n} f(e)\rho^*(n/e) = f(n)$, *then* $\rho^*(n) = \rho(n)$ *for every* $n \in Z$.

Proof: See [2, lemma 4].

Suppose now that μ^* is such that

$$(7) \quad g(n) = \sum_{\substack{d|n \\ d\in Q}} f(n/d) \Leftrightarrow f(n) = \sum_{d|n} g(d)\mu^*(n/d).$$

Then

$$f(n) = \sum_{d|n} g(d)\mu^*(n/d) = \sum_{d|n} \mu^*(n/d)\cdot \sum_{\substack{\delta e=d \\ \delta\in Q}} f(e)$$

$$= \sum_{e|n} f(e) \sum_{\substack{d\delta'=n \\ \delta e=d \\ \delta\in Q}} \mu^*(\delta') = \sum_{e|n} f(e) \sum_{\substack{\delta\delta'=n/e \\ \delta\in Q}} \mu^*(\delta');$$

writting $\rho^*(n) = \Sigma_{n=\delta d, d\in Q} \mu^*(\delta)$ we have $f(n) = \Sigma_{e|n} f(e)\rho^*(n/e)$, so by lemma 4 and theorem 1,

$$\rho(n) = \rho^*(n) = \Sigma \mu^*(\delta) \Rightarrow \mu^* = \mu_P.$$

Thus we have proved the

THEOREM 3. *Let* f *and* g *be arithmetical functions such that* $g(1) \neq 0$. *If*

$$g(n) = \sum_{\substack{d|n \\ d\in Q}} f(n/d) \iff f(n) = \sum_{d|n} g(d)\mu^*(n/d)$$

then $\mu^* = \mu_P$.

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