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REMARKS ON WEAKLY CONTINUOUS FUNCTIONS IN BANACH SPACES

por

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Let E be a Banach space over the reals and let E* be the dual space. Let = $(\alpha_1, \dots, \alpha_n)$ be a finite sequence of non-negative integers and $u = (u_1, \dots, u_n)$ a finite sequence of elements in E*. The notation $u^{\alpha} = u_1^{\alpha_1} \dots u_n^{\alpha_n}$ is standard and will used throughout. We will write $|\alpha| = \alpha_1 + \dots + \alpha_n$. Any real valued function in E of the form $P = \sum_{|\alpha| \leq n} a_{\alpha} u^{\alpha}$, a_{α} a real number, is said to be a polynomial. Clearly, every polynomial is weakly continuous.

THEOREM 1.- Let E be a reflexive Banach space and let $f : E \to R^1$. Then f is weakly continuous if and only if there is a sequence $\{P_n\}$ of polynomials that converges to f uniformly on every bounded set.

Proof. Given $x, y \in E$, $x \neq y$, there is some $u \in E^*$ such that $u(x) \neq u(y)$ (by the Hahn-Banach Theorem). Since the unit ball $\{x \} \|x\| \leq 1\}$ is weakly compact (E is reflesive), the theorem follows from the Stone-Weierstrass. Theorem see [1]; p. 281).

THEOREM 2.- Let E be an infinite dimensional Banach space over the reals, A E an open and bounded subset. Let $f : \bar{A} \to R^1$ be weakly continuous. Then $f(\bar{A}) \subset \overline{f(A)}$.

(A is the boundary of A).

Proof. It is enough to show that $\bar{A}$ is contained in the weak closure of ∂A. Let x be an interior point of A and $V(x) = A \cap \left(\bigcap_{i=1}^{n} u_i^{-1} \right)(]u_i(x) - \varepsilon, u_i(x) + \varepsilon[)$ a weak neighborhood of x in A. Since E is infinite dimensional, there is some $y \neq 0$ such that $u_i(y) = 0$ for $i = 1,2, \ldots, n$. Now, $x + ty \in \partial A$ for some t, because A is bounded and open, so $x + ty \in V \cap \partial A$.

It is proved in ([2]); p.76) that if $f : E \longrightarrow R^1$ has a compact derivative $f' : E \longrightarrow E^*$, then f is weakly continuous. The following example shows that the converse is not true.

<u>EXAMPLE</u>.- Let ℓ^2 be the usual Hilbert space of real sequences $X = (x_1, x_2, \ldots)$ such that $\sum_i x_1^2 < \infty$. Let $\Omega_r =$

$= \{ x \{ \|x\| \leq r < 1 \}$ and let $a_n = (a_1^n)$ where $a_i^n = 0$ if $i \neq n$, $a_n^n = r$. Define $f : \Omega_1 \longrightarrow R^1$ by $f(x) = \sum_{n=1}^{\infty} n^{-1} x_n^n$.

Then $\left| f(x) - \sum_{n=1}^{p} n^{-1} x_n^n \right| = \left| \sum_{n>p} n^{-1} x_n^n \right| \leq \left(\sum_{n>p} n^{-2} \right)^{\frac{1}{2}} \left(\sum_{n>p} x_n^{2n} \right)^{\frac{1}{2}}$

$\leq \sum_{n>p} n^{-2}$, since $|x_n| \leq 1$ for all n.

This shows that f is weakly continuous since it is the uniforme limit of polynomials on every Ω_r.

However, $f'(x) = \sum_{n=1}^{\infty} x_n^{-1} e_n$ is not compact since $f'(a_n) = a_n$.

<u>REMARK</u>. The example above shows that a weakly continuous function of class C^1 from an infinite dimensional Hilbert space into the reals can not be approximated, in general, by polynomials in the topology of uniform convergence of f and its derivatives on bounded sets. For suppose there is a sequence $\{P_n\}$ of polynomials such that $\{P_n\} \longrightarrow f$, $\{P_n'\} \longrightarrow f'$ uniformly in every bounded set. A trivial calculation shows that $P'(E)$ is finite dimensional for every polynomial P. This would imply that f' is compact (see [2] ; Th 1.1 p. 10).

It would be interesting to know the closure in the C^1 topology of the algebra of polynomials in an infinite dimensional Hilbert space. The finite dimensional case is the classical Bernstein's Theorem.

REFERENCES

1.- J. Dugundji "Topology, Allyn and Bacon (1966)

2.- M.M. Vainberg, "Variational Methods for the study of Nonlinear Operators," Holden Day (1964).