ON THE CONVERGENCE OF GALERKIN APPROXIMATIONS

by

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Let $X$ be a separable Banach space over the reals and let $X^*$ be its dual. If $x \in X$ and $u \in X^*$, we will write $\langle x, u \rangle$ instead of $u(x)$. Also, if $P : X \to X$ is a linear operator we will denote by $P^*$ the adjoint from $X^*$ into $X^*$ which is defined by $\langle x, P^* u \rangle = \langle P x, u \rangle$. The strong convergence in $X$ will be denoted by $x_n \to x$, the weak convergence in $X$ by $x_n \rightharpoonup x$, and the $w^*$-convergence in $X^*$ by $x_n \rightharpoonup^* x$.

We say that $X$ has property (B) if there is a sequence $\{P_n\}$ of bounded linear operators from $X$ into itself such that,

i) $P_n(X) = X_n$ is finite dimensional for every $n$.

ii) $P_n^2 = P_n$.

iii) $P_n x \to x$ for every $x \in X$ and $P_n^* u \to u$ for every $u \in X^*$.

It follows from condition ii) that $(P_n^* P_n)^2 = P_n^* P_n$, and it follows from condition i) that $P_n^* (X^*) = X_n^*$ is finite dimensional. We observe that $P_n x \to x$ for every $x \in X$ implies $P_n^* u \rightharpoonup^* u$ for every $u \in X^*$. Further, any separable Hilbert space satisfies property (B). Also, the space $c_0$ of all real sequences which converge to zero satisfies property (B). More generally, any Banach space with a biorthogonal basis (see [7; p. 64]) has property (B).

Let $X$ be a separable Banach space with property (B) and let $A : X \to X^*$ be an operator (nonlinear in general) and consider the nonlinear operator equation

$$(1) \quad A x = u, \quad u \in X^*$$

Let $A_n = P_n^* A P_n$, $u_n = P_n^* u$. Then we can write the approximating equation
which is a nonlinear operator equation in finite dimensional subspaces. We say that any
solution \( x_n \) of (2) is an \( n \)-th order Galerkin approximation of (1) and \( \{ x_n \} \) is a
Galerkin approximating sequence for the equation (1). In general, the sequence \( \{ x_n \} \)
does not converge either strongly or weakly to a solution of the equation \( Ax = u \).

In this paper we give some general conditions which assure that the weak or strong limit
points of the approximating sequence \( \{ x_n \} \) are solutions of the equation \( Ax = u \).

**THEOREM 1:** Assume that the equation (1) has a Galerkin approximating sequence
\( \{ x_n \} \). Then,

i) Any strong limit point of \( \{ x_n \} \) is a solution of the equation (1) provided \( A \) is
continous.

ii) Any weak limit point of \( \{ x_n \} \) is a solution of the equation (1) provided \( A \) is
continuous from the weak topology of \( X \) to the \( w^* \)-topology of \( X^* \).

For the proof of this theorem we need the following lemmas.

**LEMMA 1:**

i) \( y_n \to y \) implies \( P_n y_n \to y \), \( y_n, y \in X \)

ii) \( u_n \to u \) implies \( P_n u_n \to u \), \( u_n, u \in X^* \)

iii) \( y_n \rightharpoonup y \) implies \( P_n y_n \rightharpoonup y \)

iv) \( u_n \rightharpoonup u \) implies \( P_n u_n \rightharpoonup u \)

**PROOF.**

i) By the principle of uniform boundedness the set \( \{ \| P_n \| \} \) is bounded,
so \( \| y - P_n y_n \| \leq \| y - P_n y \| + \| P_n y - P_n y_n \| \to 0 \) if \( n \) tends to infinity.

ii) For any \( u \in X^* \) we have

\[ \langle P_n y_n, u \rangle = \langle y_n, P_n^* u \rangle = \langle y_n, u \rangle + \langle y_n, P_n^* u - u \rangle \to 0 \] if \( n \) tends to infinity. The proof of ii) is similar to the proof of i), and so are the proofs of iii) and iv).
LEMMA 2: If $A$ is continuous, then $A_n y \to Ay$ for every $y \in X$.

**Proof.** Since $P_n y \to y$, it follows that $AP_n \to Ay$.

Thus $P_n^* A P_n y = A_n y \to Ay$ by lemma 1.

**Proof of Theorem 1.** i) If $x$ is a strong limit point of $x_n$ there is a subsequence $\{x_{n_k}\}$ which converges to $x$, so $P_{n_k}^* A x_{n_k} \to Ax$ by lemma 1 (part ii).

Since $P_{n_k}^* A x_{n_k} = u_{n_k} \to u$, it follows that $Ax = u$.

ii) Let $x_{n_k} \overset{w}{\to} x$. Then one uses part iv) of lemma 1 to show that $Ax = u$.

**Theorem 2:** We make the following assumptions:

i) Let $D(A)$ be the domain of the operator $A$ from $X$ into $X^*$. Let $y_n \overset{w}{\to} y$ and $Ay_n \overset{w^*}{\to} v$. Then $y \in D(A)$ and $Ay = v$.

ii) There is a Galerkin approximating sequence $\{x_n\}$ for the equation $Ax = u$ such that $\{A x_n\}$ is bounded.

Then any weak limit point of $\{x_n\}$ is a solution of the equation (1).

The proof of this theorem is based on the following lemma:

**Lemma 3:** Assume that the equation (1) has a Galerkin approximating sequence $\{x_n\}$. Then $\{A x_n\}$ is bounded if and only if $A x_n \overset{w^*}{\to} u$ (we do not assume that $A$ is continuous).

**Proof.** Let $z \in X$ and assume $\{A x_n\}$ is bounded. Then,

$$<z, Ax_n> = <z - P_n z, A x_n> + <P_n z, A x_n>$$

Now, $<z - P_n z, A x_n> \to 0$ and $<P_n z, A x_n> = <z, P_n^* A x_n> = <z, u_n> \to <z, u>$. Therefore $Ax_n \overset{w^*}{\to} u$. If $Ax_n \overset{w^*}{\to} u$ then $\{A x_n\}$ is $w^*$ bounded, so it is norm bounded by a well-known theorem.

**Proof of Theorem 2.** Let $x$ be a weak limit point of $\{x_n\}$ and let $x_{n_k} \overset{w}{\to} x$. 

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Then \( A x \xrightarrow{w^*} u \) by lemma 3 and condition ii), so \( A x = u \) by condition i).

**Theorem 3.** The hypothesis of theorem 2. Then,

a) If \( A \) has an inverse \( A^{-1} \) which is continuous from the \( w^* \) - topology into the strong (weak) topology, then \( \{ x_n \} \) converges strongly (weakly) to the unique solution of the equation (1).

b) If \( A^{-1} \) is compact, then \( \{ x_n \} \) converges strongly to the unique solution of equation (1).

**Proof.** a) Let \( A x_n = v_n \). Then \( v_n \xrightarrow{w^*} u \) by lemma 3, so \( x_n = A^{-1}(v_n) \) converges strongly (weakly), say, to \( x \in X \). By condition ii) \( x \in D(A) \) and \( A x = u \).

b) Since \( A^{-1} \) is compact, it follows that \( \{ x_n \} = \{ A^{-1} v_n \} \) has a strong limit point. If \( x \) is a limit point of \( x_n \), then \( A x = u \) by theorem 2.

**Remark.** The conditions given in the previous theorem can be applied to unbounded linear operators. If \( A \) is a differential operator, then the domain \( D(A) \) of \( A \) is some dense subspace of a Hilbert space \( H \). If we assume that \( A \) is symmetric and satisfies the condition

\[
< A x, x > \geq c \| x \|^2, \quad c > 0, \quad x \in D(A)
\]

then \( A \) can be extended to an operator \( B[ (D(B) \supset D(A)) \) and the restriction of \( B \) to \( D(A) \) is \( A \) such that its range is all of \( X \) and \( B^{-1} \) is continuous. Usually \( B \) is referred to as the Friedrichs extension of \( A \). For details see [4; p. 5]. We observe also that in many boundary value problems the inverse operator \( A^{-1} \) is an integral operator defined by a Green function, so in many cases \( A^{-1} \) is compact. Since

\[
< A_n x_n, x_n > = < p_n A x_n, x_n > = < A x_n, x_n > \geq c \| x_n \|^2
\]
it follows that \( A_n : X_n \to X_n \) is one to one and linear, so the range of \( A_n \) is \( x_n \). Therefore (3) implies the existence of a Galerkin approximating sequence \( \{ x_n \} \) for every \( u \in X \). In general, however, \( \{ A x_n \} \) is not bounded.

Let \( A \) be a strongly monotone map from \( X \) into \( X^* \), i.e. \( A \) satisfies

\[
< x - y, A x - A y > \geq c \| x - y \|^2
\]

for every \( x, y \in D(A) \) and some constant \( c > 0 \). Then it follows from (4) that

\[
\| A_n x - A_n y \| \geq c \| x - y \|, x, y \in X_n.
\]

A simple argument based on Brouwer's theorem on invariance of domain (see Browder [1] or Petryshyn [6]) shows that \( A_n x_n = u_n \) has a unique solution which is denoted by \( x_n \). Therefore, if \( A \) is strongly monotone and \( X_n \subset D(A) \) for every \( n \), then one can show the existence of a Galerkin approximating sequence. Moreover, \( \{ x_n \} \) converges strongly to the unique solution of the equation \( A x = u \) if \( A \) is strongly monotone and continuous, as has been shown by the authors mentioned before. We now state a theorem for noncontinuous, strongly monotone operators. We assume that \( A \) satisfies

\[
\| A x \| \leq M \| x \|
\]

for all \( x \in X_n \) and some constant \( M \) which does not depend upon \( n \).

**Theorem 4**: Let \( X \) be a reflexive Banach space with property \( (B) \) and let \( A \) be a densely defined operator from \( X \) into \( X^* \). Assume that \( A \) satisfies (4) and (6) and \( A^{-1} \) is continuous from the \( w_\ast \)-topology into the strong (weak) topology. Then equation (2) has a unique solution \( x_n \) for each \( n \), and \( \{ x_n \} \) converge strongly (weakly) to the unique solution of equation (1).

**Proof**. The operator \( A_n \) maps \( X_n \) into \( X_n^* \) and (5) implies that
$A_n$ is one-to-one. Also, (5) implies that the range of $A_n$ is closed. By Brouwer's theorem on invariance of domain, the range of $A_n$ is open. Therefore the range of $A_n$ is $x^*_n$ since it is connected. Thus, we have shown that the equation $A_n x_n = u_n$ has a unique solution. Next, we show that $\{x_n\}$ is bounded. From (5), putting $y = 0$, we obtain

$$c || x_n || \leq || A_n(x_n) - A_n(0) || \leq || u_n || + || A_n(0) || \leq N$$

for some constant $N$ since $u_n \to u$ and $A_n(0) \to A(0)$. Now, (6) implies that $\{A x_n\}$ is bounded, so by lemma 3 $A x_n u^* \to u$. Since $A^{-1} u$ is defined, it follows that $\{x_n\}$ converges strongly (weakly) to $A^{-1} u = x$. This concludes the proof.

REFERENCES

7. VAINBERG, M. M., "Variational methods for the study of monlinear operators", Holden-Day (1964)

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