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ON THE CONVERGENCE OF GALERKIN APPROXIMATIONS

by

Guillermo RESTREPO

Let X be a separable Banach space over the reals and let X^* be its dual. If $x \in X$ and $u \in X^*$ we will write $<x, u>$ instead of $u(x)$. Also, if $P : X \to X$ is a linear operator we will denote by P^* the adjoint from X^* into X^* which is defined by $<x, P^* u> = <P x, u>$. The strong convergence in X will be denoted by $x_n \to x$, the weak convergence in X by $x_n \overset{w}{\to} x$ and the w^* - convergence in X^* by $x_n \overset{w^*}{\to} x$. We sat that X has property (B) if there is a sequence $\{ P_n \}$ of bounded linear operator from X into itself such that,

i) $P_n(X) = X_n$ is finite dimensional for very n.

ii) $P_n^2 = P_n$

iii) $P_n x \to x$ for every $x \in X$ and $P_n^* u \to u$ for every $u \in X^*$.

It follows from condition ii) that $(P_n^*)^2 = P_n^*$, and it follows from condition i) that $P_n^*(X^*) = X_n^*$ is finite dimensional. We observe that $P_n x \to x$ for every $x \in X$ implies $P_n^* u \overset{w^*}{\to} u$ for every $u \in X^*$. Further, any separable Hilbert space satisfies property (B). Also, the space c_0 of all real sequences which converge to zero satisfies property (B). More generally, any Banach space with a biorthogonal basis (see [7 ; p. 64] has property (B).

Let X be a separable Banach space with property (B) and let $A : X \to X^*$ be an operator (nonlinear in general) and consider the nonlinear operator equation

(1) $A x = u, \ u \in X^*$

Let $A_n = P_n^* A P_n$, $u_n = P_n^* u$. Then we can write the approximating equation

$$(2) \qquad A_n x_n = u_n , \; x_n \in X_n , \; u_n \in X_n^*$$

which is a nonlinear operator equation in finite dimensional subspaces. We say that any solution x_n of *(2)* is an <u>n-th order Galerkin approximation of *(1)*</u> and $\{ x_n \}$ is a <u>Galerkin approximating sequence</u> for the equation *(1)*. In general, the sequence $\{ x_n \}$ does not converge either strongly or weakly to a solution of the equation $A x = u$.

In this paper we give some general conditions which assure that the weak or strong limit points of the approximating sequence $\{ x_n \}$ are solutions of the equation $Ax = u$.

<u>*THEOREM 1:*</u> Assume that the equation *(1)* has a Galerkin approximating sequence $\{ x_n \}$. Then,

i) Any strong limit point of $\{ x_n \}$ is a solution of the equation *(1)* provided A is continuos.

ii) Any weak limit point of $\{ x_n \}$ is a solution of the equation *(1)* provided A is continuous from the weak topology of X to the w^* - topology of X^*.

For the proof of this theorem we need the following lemmas.

<u>*LEMMA 1:*</u> i) $y_n \to y$ implies $P_n y_n \to y$, $y_n \in X$, $y \in X$

ii) $u_n \to u$ implies $P_n^* u_n \to u$, $u_n \in X^*$, $u \in X^*$

iii) $y_n \overset{w}{\to} y$ implies $P_n y_n \overset{w}{\to} y$

iv) $u_n \overset{w^*}{\to} u$ implies $P_n^* u_n \overset{w^*}{\to} u$

<u>*PROOF.*</u> i) By the principle of uniform boundedness the set $\{ \| P_n \| \}$ is bounded, so $\| y - P_n y_n \| \leqslant \| y - P_n y \| + \| P_n y - P_n y_n \| \to 0$ if n tends to infinity.

ii) For any $u \in X^*$ we have

$< P_n y_n , u > = < y_n , P_n^* u > = < y_n , u > + < y_n , P_n^* u - u > \to 0$ if n tends to infinity. The proof of ii) is similar to the proof of i), and so are the proofs of iii) and iv).

LEMMA 2 : If A is continuous, then $A_n y \to Ay$ for every $y \epsilon X$.

PROOF . Since $P_n y \to y$, it follows that $AP_n \to Ay$.

Thus $P_n^* A P_n y = A_n y \to Ay$ by lemma *1*.

PROOF OF THEOREM 1. i) If x is a strong limit point of x_n there is a subsequence $\{x_{n_\kappa}\}$ which converges to x, so $P_{n_k}^* A x_{n_k} \to Ax$ by lemma *1* (part ii).

Since $P_{n_k}^* A x_{n_k} = u_{n_k} \to u$, it follows that $Ax = u$.

ii) Let $x_{n_k} \overset{w}{\to} x$. Then one uses part iv) of lemma *1* to show that $Ax = u$.

THEOREM 2 : We make the following assumptions :

i) Let $D(A)$ be the domain of the operator A from X into X^* , let $y_n \overset{w}{\to} y$ and $Ay_n \overset{w^*}{\to} v$. Then $y \epsilon D(A)$ and $Ay = v$.

ii) There is a Galerkin approximating sequence $\{x_n\}$ for the equation $Ax = u$ such that $\{A x_n\}$ is bounded.

Then any weak limit point of $\{x_n\}$ is a solution of the equation *(1)* .

The proof of this theorem is based on the following lemma :

LEMMA 3 : Assume that the equation *(1)* has a Galerkin approximating sequence $\{x_n\}$. Then $\{A x_n\}$ is bounded if and only if $A x_n \overset{w^*}{\to} u$ (we do not assume that A is continuous).

PROOF . Let $z \epsilon X$ and assume $\{A x_n\}$ is bounded. Then,

$$< z , A x_n > = < z - P_n z, A x_n > + < P_n z, A x_n >$$

Now, $< z - P_n z, A x_n > \to o$ and $< P_n z, A x_n > = < z, P_n^* A x_n > = < z, u_n > \to < z, u >$.

Therefore $A x_n \overset{w^*}{\to} u$. If $A x_n \overset{w^*}{\to} u$ then $\{A x_n\}$ is w^* , bounded, so it is norm bounded by a well - known theorem .

PROOF OF THEOREM 2. Let x be a weak limit point of $\{x_n\}$ and let $x_{n_k} \overset{w}{\to} x$.

Then $A x_{n_k} \overset{w^*}{\to} u$ by lemma *3* and condition ii), so $A x = u$ by condition i).

THEOREM 3. The hypothesis of teorem *2*. Then,

a) If A has an inverse A^{-1} which is continuous from the w^* - topology into the strong (weak) topology, then $\{ x_n \}$ converges strongly (weakly) to the unique solution of the equation *(1)*

b) If A^{-1} is compact, then $\{ x_n \}$ converges strongly to the unique solution of equation *(1)*.

PROOF. a) Let $A x_n = v_n$. Then $v_n \overset{w^*}{\to} u$ by lemma *3*, so $x_n = A^{-1}(v_n)$ converges strongly (weakly), say, to $x \in X$. By condition ii) $x \in D(A)$ and $A x = u$

b) Since A^{-1} is compact, it follows that $\{ x_n \} = \{ A^{-1} v_n \}$ has a strong limit point. If x is a limit point of x_n, then $A x = u$ by theorem *2*.

REMARK. The conditions given in the previous theorem can be applied to unbounded linear operators. If A is a differential operator, then the domain $D(A)$ of A is some dense subspace of a Hilbert space H. If we assume that A is symmetric and satisfies the condition

(3) $\langle A x, x \rangle \geq c \, || x ||^2 , \quad c > 0, \quad x \in D(A)$

then A can be extended to an operator $B[(D(B) \supseteq D(A)$ and the restriction of B to $D(A)$ is $A)]$ such that its range is all of X and B^{-1} is continuous. Usually B is referred to as the Friedrichs extension of A. For details see [4; p.5]. We observe also that in many boundary value problems the inverse operator A^{-1} is an integral operator defined by a Green function, so in many cases A^{-1} is compact. Since

$$\langle A_n x_n , x_n \rangle = \langle P_n^* A x_n , x_n \rangle = \langle A x_n , x_n \rangle \geq c \, || x_n ||^2 ,$$

it follows that $A_n : X_n \to X_n$ is one to one and linear, so the range of A_n is X_n. Therefore (3) implies the existence of a Galerkin approximating sequence $\{x_n\}$ for every $u \in X$. In general, however, $\{A x_n\}$ is not bounded.

Let A be a strongly monotone map from X into X^*, i.e A satisfies

(4) $\quad < x - y , A x - A y > \geqslant c \, || x - y ||^2$

for every $x, y \in D(A)$ and some constant $c > 0$. Then it follows from (4) that

(5) $\quad || A_n x - A_n y || \geqslant c \, || x - y || , x, y \in X_n .$

A simple argument based on Brouwer's theorem on invariance of domain (see Browder [*1*] or Petryshyn [*6*]) shows that $A_n x_n = u_n$ has a unique solution which is denoted by x_n. Therefore, if A is strongly monotone and $X_n \subset D(A)$ for every n, then one can show the existence of a Galerkin approximating sequence. Moreover, $\{x_n\}$ converges strongly to the unique solution of the equation $A x = u$ if A is strongly monotone and continuous, as has been shown by the authors mentioned before. We now state a theorem for noncontinuous, strongly monotone operators. We assume that A satisfies.

(6) $\quad || A x || \leqslant M \, || x ||$

For all $x \in X_n$ and some constant M which does not depend upon n.

THEOREM 4 : Let X be a reflexive Banach space with property *(B)* and let A be a densely defined operator from X into X^*. Assume that A satisfies (4) and (6) and A^{-1} is continuous from the w^* - topology into the strong (weak) topology. Then equation (2) has a unique solution x_n for each n, and $\{x_n\}$ converge strongly (weakly) to the unique solution of equation (1).

PROOF . The operator A_n maps X_n into X_n^* and (5) implies that

A_n is one-to-one. Also, (5) implies that the range of A_n is closed. By Brouwer's theorem on invariance of domain, the range of A_n is open. Therefore the range of A_n is X_n^* since it is connected. Thus, we have shown that the equation $A_n x_n = u_n$ has a unique solution. Next, we show that $\{ x_n \}$ is bounded. From (5), putting $y = 0$, we obtain

$$c \, || x_n || \leq || A_n (x_n) - A_n (0) || \leq || u_n || + || A_n (0) || \leq N$$

for some constant N since $u_n \to u$ and $A_n (0) \to A(0)$. Now, (6) implies that $\{ A x_n \}$ is bounded, so by lemma 3 $A x_n \overset{w^*}{\to} u$. Since $A^{-1} u$ is defined, it follows that $\{ x_n \}$ converges strongly (weakly) to $A^{-1} u = x$. This concludes the proof.

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University of Puerto Rico

Mayaguez, Puerto Rico