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AN ISOMORPHISM THEOREM FOR ALGEBRAS

by

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Suppose we have given two algebras $\mathfrak{A}$ and $\mathfrak{B}$ over a field F and an F-linear ϕ such that

$$\phi \quad : \mathfrak{A} \to \mathfrak{B} \quad \text{such that } \phi|_{\mathfrak{A}^2} : \mathfrak{A}^2 \to \mathfrak{B}^2 \text{ is onto,}$$

and such that ϕ is an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as vector spaces over F. We wish to find necessary and sufficient conditions that ϕ be an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as algebras.

For any algebra $\mathfrak{A}$ the multiplication in $\mathfrak{A}$ is an F-bilibear map,

$$\tilde{\tau}_{\mathfrak{A}} : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}^2$$

Associated to τ is an F-linear map,

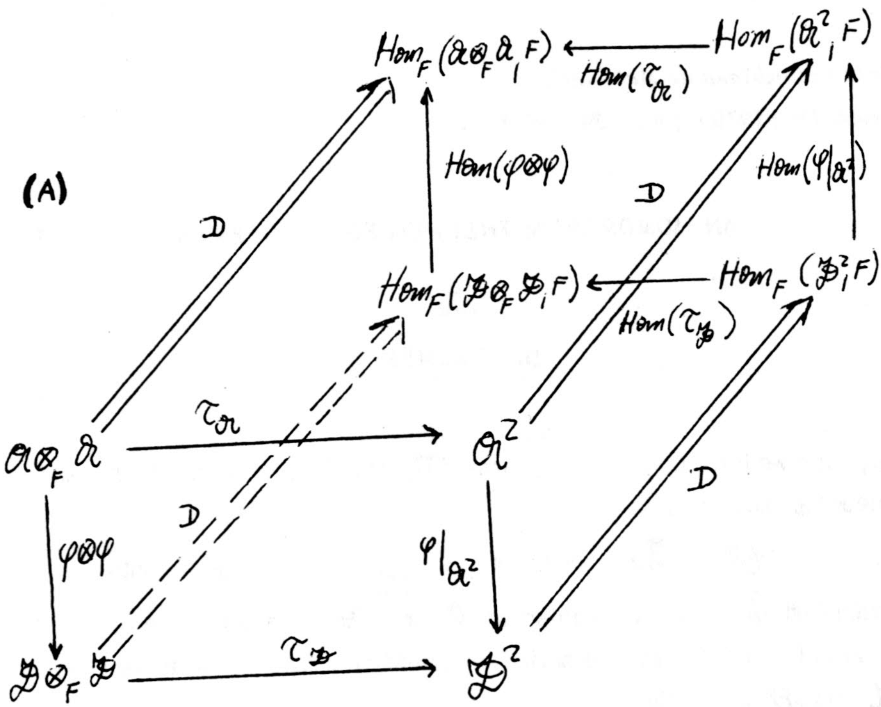
$$\tau_{\mathfrak{A}} : \mathfrak{A} \otimes_F \mathfrak{A} \longrightarrow \mathfrak{A}^2$$

It is immediate from the definitions that ϕ will be an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as algebras if and only if the following diagram commutes:

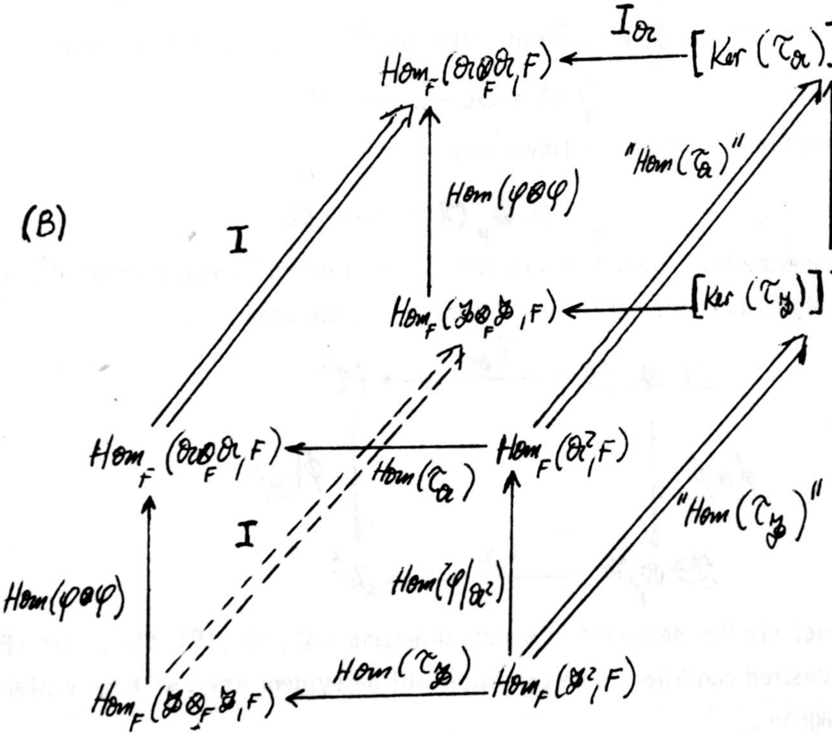
$$\begin{array}{ccc} \mathfrak{A} \otimes_F \mathfrak{A} & \xrightarrow{\ \tau_{\mathfrak{A}}\ } & \mathfrak{A}^2 \\ \downarrow {\scriptstyle \phi \otimes \phi} & & \downarrow {\scriptstyle \phi|_{\mathfrak{A}^2}} \\ \mathfrak{B} \otimes_F \mathfrak{B} & \xrightarrow{\ \tau_{\mathfrak{B}}\ } & \mathfrak{B}^2 \end{array}$$

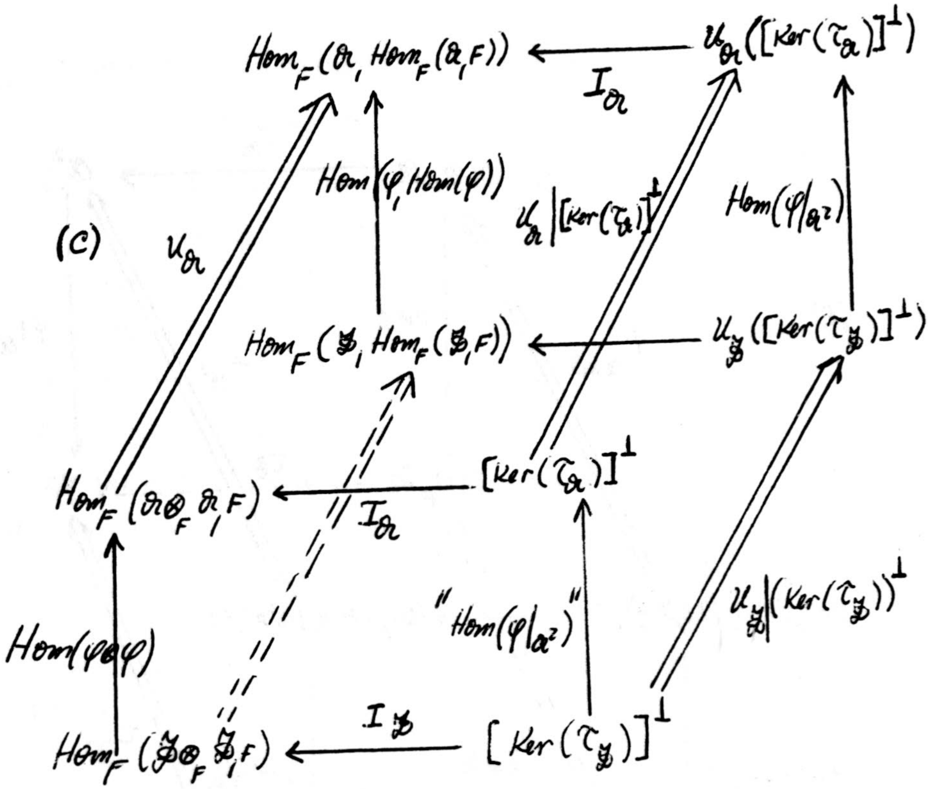
We proceed via the sequence of square diagrams (A), (B), (C), (D), and (E) to get the desired condition. The condition will be evident when we have explained each diagram.

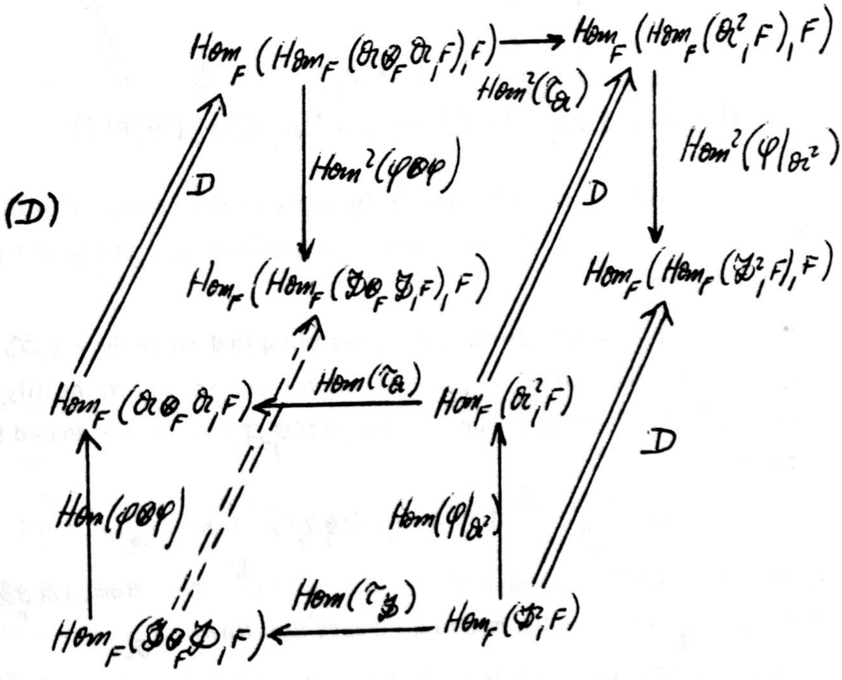
(A)



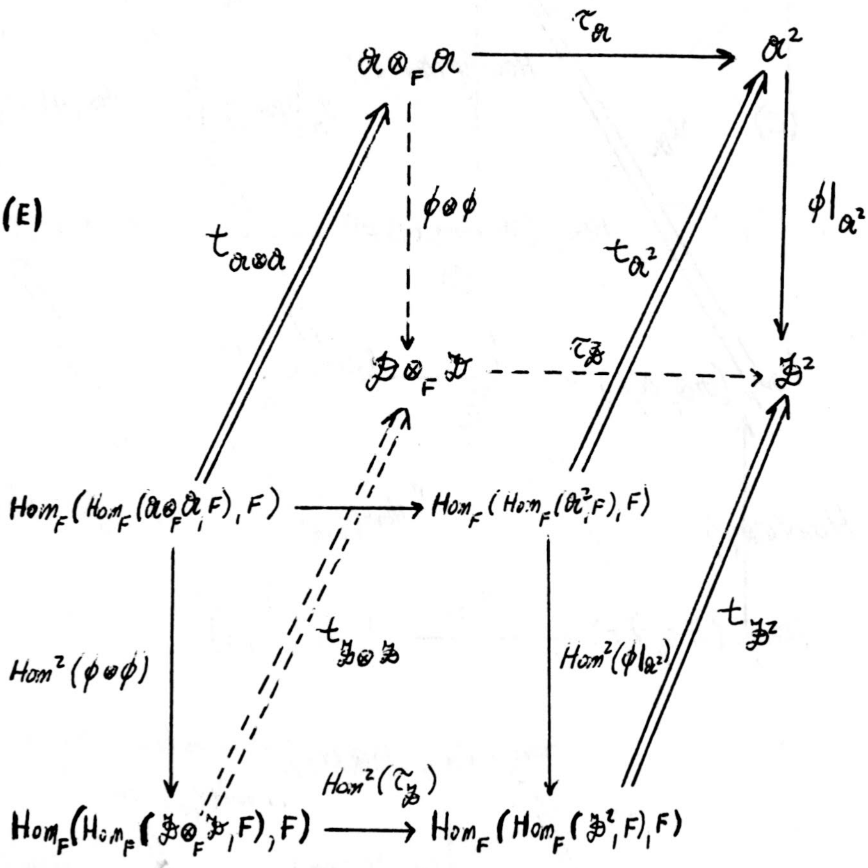
(B)




(C)
Hom_F(𝔄, Hom_F(𝔄, F))
𝒰_𝔄([Ker(τ_𝔄)]^⊥)
I_𝔄
Hom(φ, Hom(φ))
𝒰_𝔄|[Ker(τ_𝔄)]^⊥
Hom(φ|𝔄²)
𝒰_𝔄
Hom_F(𝔅, Hom_F(𝔅, F))
𝒰_𝔅([Ker(τ_𝔅)]^⊥)
Hom_F(𝔄⊗_F 𝔄, F)
[Ker(τ_𝔄)]^⊥
I_𝔄
"Hom(φ|𝔄²)"
𝒰_𝔅|(Ker(τ_𝔅))^⊥
Hom(φ⊗φ)
I_𝔅
Hom_F(𝔅⊗_F 𝔅, F)
[Ker(τ_𝔅)]^⊥


(D)
Hom_F(Hom_F(𝔄⊗_F 𝔄, F), F)
Hom_F(Hom_F(𝔄², F), F)
Hom²(τ_𝔄)
Hom²(φ⊗φ)
Hom²(φ|𝔄²)
D
Hom_F(Hom_F(𝔅⊗_F 𝔅, F), F)
Hom_F(Hom_F(𝔅², F), F)
Hom(τ_𝔄)
Hom_F(𝔄⊗_F 𝔄, F)
Hom_F(𝔄², F)
Hom(φ⊗φ)
Hom(φ|𝔄²)
Hom(τ_𝔅)
Hom_F(𝔅⊗_F 𝔅, F)
Hom_F(𝔅², F)

(E)



(A) : In (A) we get (2) from (1) by applying the functor $D = Hom_F(\ \ , F)$ which assigns to every F-linear space its algebraic dual and to every F-linear map its adjoint.

(B) : In (B) we get (3) from (2) by recalling that since for any $\mathfrak{A}$, $\tau_{\mathfrak{A}}$ is surjective, we have $Hom_F(\tau_{\mathfrak{A}})$ injective; and so we can identify $Hom_F(\mathfrak{A}^2, F)$ with its image in $Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F)$. It is standard that this image is,

$$[Ker(\tau_{\mathfrak{A}})]^{\perp} = \{ f \in Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F) \mid f(Ker(\tau_{\mathfrak{A}})) = 0 \}$$

We let $I_{\mathfrak{A}}$ be the inclusion of $[Ker(\tau_{\mathfrak{A}})]^{\perp}$ into $Hom_F(\mathfrak{A}\otimes_F\mathfrak{A}, F)$ and "$Hom_F(\tau_{\mathfrak{A}})$" be the mapping induced by $Hom_F(\tau_{\mathfrak{A}})$.

(C) : In (C) we get (4) from (3) by recalling that there is a natural isomorphism

$u_{(\bullet)}$ between the following two functors of one variable,

$$Hom_F(\bullet \otimes_F \bullet, F) \quad \text{and} \quad Hom_F(\bullet, Hom_F(\bullet, F)).$$

(D) In (C) we get (5) from (2) by applying the functor D again.

(E) In (E) we get (1) from (5) by $t_{(\bullet)}$ where $t_{(\bullet)}$ is the natural isomorphism between the two functors D^2 and I.

REMARK. We must at this point assume some such condition as $dim < \infty$ so that D^2 will be naturally isomorphic to I.

We can now state the main theorem.

THEOREM 1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finite-dimensional algebras over F. Suppose we have an F-linear ϕ such that

$$\phi : \mathfrak{A} \rightarrow \mathfrak{B} \ \text{ such that } \ \phi|_{\mathfrak{A}^2} : \mathfrak{A}^2 \rightarrow \mathfrak{B} \ \text{ is onto}$$

and such that ϕ is a vector space isomorphism. Then ϕ will be an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as algebras <=> the following diagram, denoted above by (4), commutes:

$$\begin{array}{ccc}
Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F)) & \xleftarrow{\ I_{\mathfrak{A}}\ } & u_{\mathfrak{A}}([Ker(\tau_{\mathfrak{A}})]) \\
\Big\uparrow {\scriptstyle Hom(\phi,\, Hom(\phi))} & & \Big\uparrow {\scriptstyle \text{“}Hom(\phi|_{\mathfrak{A}^2})\text{”}} \\
Hom_F(\mathfrak{B}, Hom_F(\mathfrak{B}, F)) & \xleftarrow{\ I_{\mathfrak{B}}\ } & u_{\mathfrak{B}}([Ker(\tau_{\mathfrak{B}})])
\end{array}$$

PROOF: Since we have seen that ϕ is an algebra isomorphism <=> (1) commutes, the proof just consists of the fact that as soon as any one of the squares (1) through (5) commute, they all do.

Use the notation "$(k) \Rightarrow (l)$" for (k) commutes $\Rightarrow (l)$ does.

(1) => (2): This follows from the functorial property of D.

(2) <=> (3): The faces bounded (in part) by the double edges in (B) commute. Since I, "$Hom(\tau_{\mathfrak{A}})$" and "$Hom(\tau_{\mathfrak{B}})$" are isomorphism, (2) <=> (3). (3) <=> 4): The faces bounded (in part) by the double edges of (C) commute since $u_{(\bullet)}$ is a natural isomorphism. Again since $u_{(\bullet)}$ is an isomorphism, (3) <=> (4). (2) => (5): Again the functorial property of D.

(5) < = > (1) : The faces bounded (in part) by the double edges in (E) commute since $t_{(\bullet)}$ is an isomorphism (5) < = > (1) . Since $t_{(\bullet)}$ is an isomorphism (5) < = > (1) .

REMARK . The essential feature of the whole proof is that D^2 is naturally isomorphic to I . (This gives us the "if" part .) We could get other necessary and sufficient conditions for isomorphism if we had other invertible functors i.e functors F for which there is a functor G such that $G \circ F$ is naturally isomorphic to the identity functor .

It would be rice to have a necessary and sufficient isomorphism condition that explicitly shows the involvement of the base field and so , we will now choose bases for our F- spaces .

Let $\{ a_i \}^n$, and $\{ \beta_i \}^n$, be bases for and respectively , and such that $\{ a_i \}^r$, and $\{ \beta_i \}^r$, are bases for and respectively . Thus we have ,

$$\begin{vmatrix} a_1 a_1 & \cdots & a_1 a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n a_1 & \cdots & a_n a_n \end{vmatrix} = \sum_{k=1}^{n} \begin{vmatrix} c_{11}^k & \cdots & c_{1n}^k \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ c_{n1}^k & & c_{nn}^k \end{vmatrix} a_k$$

, and ,

$$\begin{vmatrix} \beta_1 \beta_1 & \cdots & \beta_1 \beta_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \beta_n \beta_1 & \cdots & \beta_n \beta_n \end{vmatrix} = \sum_{k=1}^{r} \begin{vmatrix} d_{11}^k & \cdots & d_{1n}^k \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ d_{n1}^k & \cdots & d_{nn}^k \end{vmatrix} \beta_k$$

The $\{ c_{ij}^k \}$ and $\{ d_{ij}^k \}$ are the multiplication constants of and with respect to the given bases . Let us write , $C^k = [c_{ij}^k]$ and $D^k = [d_{ij}^k]$. Further let $H = [h_{ij}]$ be the matrix for ϕ relative to the given bases .

NOTE : In the present notation the matrix for $\phi \mid_{\mathfrak{a}^2}$ is the upper - left hand corner of H . $\bar{H}$ has the form ;

$$(7)\quad \left[\begin{array}{ccc|ccc} b_{11} & \cdots & b_{1r} & & & \\ \vdots & & \vdots & & & \\ b_{r1} & \cdots & b_{rr} & & & \\ \hline b_{r+1\,1} & \cdots & & b_{r+1\,r+1} & \cdots & \\ \vdots & & & \vdots & \ddots & \\ \vdots & & & \vdots & & b_{nn} \end{array}\right] \text{ where } H|_{\mathfrak{A}^2} = \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & & \vdots \\ b_{r1} & \cdots & b_{rr} \end{bmatrix}$$

We are now ready to state theorem (1) in the language of matrices.

THEOREM 2. - Let $\mathfrak{A}$ and $\mathfrak{B}$ be n-dimensional algebras over F. Suppose $\mathfrak{A}^2$ and $\mathfrak{B}^2$ are both r-dimensional. Further suppose we have an F-linear ϕ such that

$$\phi : \mathfrak{A} \to \mathfrak{B} \quad \text{and} \quad \phi|_{\mathfrak{A}^2} : \mathfrak{A}^2 \to \mathfrak{B}^2$$

and such that ϕ is a vector space isomorphism. Then ϕ will be an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$ as algebras $<=>$ the following set of matrix equations holds,

$$\begin{aligned} HD^1 H^T &= b_{11} C^1 + \dots + b_{r1} C^r \\ &\;\vdots \\ HD^r H^T &= b_{1r} C^1 + \dots + b_{rr} C^r \end{aligned}$$

That is,

$$(8)\quad \begin{bmatrix} HD^1 H^T \\ \vdots \\ HD^r H^T \end{bmatrix} = H|_{\mathfrak{A}^2}^{T} \begin{bmatrix} C^1 \\ \vdots \\ C^r \end{bmatrix}$$

PROOF : For any algebra $\mathfrak{A}$, both $Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ and $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ will be isomorphic to a space of $n \times n$ F-matrices. Denote these by $Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ and $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F)$ respectively. The proof will now consist mainly of three observations.

(i) The natural isomorphism between $Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ and $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ preserves matrices, i.e., if to $f \in Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ we associate the matrix M_f then $t_{\mathfrak{A}}(f) \in Hom_F(\quad, F))$ has the same matrix M_f associated to it (provided we use the dual basis for $Hom_F(\mathfrak{A}, F))$. To see why the last statment is so, recall that every $f \in Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ arises in the following way:

$$f(a_i \otimes a_j) = v_1(a_i) \quad v_2(a_j)$$

where $v_1, v_2 \in Hom_F(\mathfrak{A}, F)$. This correspondence between pairs of v_1, v_2 in $Hom_F(\mathfrak{A}, F)$ and $f \in Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ is one-to-one. We thus write $f = f_{v_1, v_2}$.

The natural isomorphism $t_{\mathfrak{A}}$ from $Hom_F(\mathfrak{A} \otimes_F \mathfrak{A}, F)$ to $Hom_F(\mathfrak{A}, Hom_F(\mathfrak{A}, F))$ is then defined by,

$$f_{v_1, v_2} \rightarrow S_{f_{v_1, v_2}}(a) = v_1(a) \; v \quad \text{for } a \in \mathfrak{A}.$$

If $\{a_i^*\}^n \subset Hom_F(\mathfrak{A}, F)$ is the dual base to $\{a_i\}^n$, we have $v_2 = v_2(a_1)\, a_1^* + \ldots + v_2(a_n)\, a_n^*$.

Thus,

$$S_{f_{v_1, v_2}}(a_1) = v_1(a_1)\, v_2(a_1)\, a_1^* + \ldots + v_1(a_1)\, v_2(a_n)\, a_n^*$$

$$\vdots$$

$$S_{f_{v_1, v_2}}(a_n) = v_1(a_n)\, v_2(a_1)\, a_1^* + \ldots + v_1(a_n)\, v_2(a_n)\, a_n^*$$

And so we see that the matrix for the linear transformation $S_{f_{v_1, v_2}}$ is the same as that for the functional f_{v_1, v_2}.

(ii) For any algebra $\mathfrak{A}$, the mapping $\tau_{\mathfrak{A}} : \mathfrak{A} \otimes_F \mathfrak{A} \rightarrow \mathfrak{A}^2$ defined above has r component functionals with respect to the base $\{a_i\}_1^r$ of $\mathfrak{A}^2$. Clearly, $\tau_{\mathfrak{A}}^k \in [Ker(\tau_{\mathfrak{A}})]^{\perp}$ for $k = 1, \ldots, r$. Indeed, upon checking the isomorphism between $[Ker(\tau_{\mathfrak{A}})]^{\perp}$ and $Hom_F(\mathfrak{A}^2, F)$ one sees that $\{\tau^k_{\mathfrak{A}}\}_1^r$, corresponds to the dual base $\{a_i^*\}_1^r$ of $Hom_F(\mathfrak{A}^2, F)$. By its definition, the matrix C^k is the matrix for τ^k. Let $<\{C^k\}>$ denote the F-space generated by the $\{C^k\}$.

(iii) The mapping,

"$Hom_F(\phi\,,\,Hom_F(\phi\,))$" : $Hom_F(\mathfrak{D}\,,\,Hom_F(\mathfrak{D},\,F\,)) \rightarrow Hom_F(\mathfrak{A}\,,\,Hom_F(\mathfrak{A},\,F))$ induced by $Hom_F(\phi\,,\,Hom_F(\phi\,))$ is given by,

$$M_S \rightarrow HM_S H^T \;.$$

This follows directly from the definition of $Hom_F(\phi\,,\,Hom_F(\,\phi\,))\,:\,S \rightarrow Hom\,(\phi\,)\;o\;S\;o\;\phi$ and the fact that the matrices for $Hom_F(\phi\,)$ and $Hom_F(\phi\,)\;o\;S\;o\;\phi$ are H^T and $HM_S H^T$, respectively.

With the observations out of the way let us now consider the diagram (4) of Theorem (1) as a diagram of F - spaces of matrices.

We will then have that ϕ is an isomorphism of algebras <=> the following diagram commutes:

$$\begin{array}{ccc}
Hom_F(\mathfrak{A}\,,\,Hom_F(\mathfrak{A}\,,\,F\,)) & \xleftarrow{\quad I_C \quad} & <\{\,C^k\,\}> \\
\Big\uparrow {\scriptstyle H\,(\,.\,)\,H^T} & & \Big\uparrow {\scriptstyle H\,|^T_{\mathfrak{A}^2}} \\
Hom_F(\mathfrak{D}\,,\,Hom_F(\mathfrak{D}\,,\,F\,)) & \xleftarrow{\quad I_D \quad} & <\{\,D^k\,\}>
\end{array}$$

But the commutation is just the equation (8). ∎

For the reader who is not too sure about all these diagrams, I will now include a short computational derivation of the equation (8). Although this second proof is straight-forward, it does not yield as much insight into "what is going on" as the first proof does. For instance; the equation (8) indicates that the F - linear map $H\,(\,.\,)\,H^T : <\{\,D^k\,\}> \rightarrow <\{\,C^k\,\}>$ is an adjoint, but there is no way of seeing what duality is with out the first derivation.

PROOF : (Alternate of theorem 2)). Since $\phi\,(a_i) = \sum_{j=1}^{n} b_{ij}\,\beta_j$ multiplying out gives,

$$[\;\phi\,(a_i)\;\;.\;\;\phi\,(a_j)\,] \;=\; H\,[\;\beta_i \quad \beta_j\;]\;H^T$$

Using the fact that $H(\,.\,)\,H^T$ acts linearly on F - linear combinations of matrices we then have,

$$(*)\qquad [\phi(a_i)\,.\,\phi(a_j)] = H\,[\beta_i\ \beta_j]\,H^T = \sum_{k=1}^{r} H D^k H^T \beta_k .$$

On the other hand, $\phi(a_i) = \sum_{j=1}^{r} b_{ij}\,\beta_j$ for $i = 1, \ldots, r$ gives,

$$(**)\qquad \sum_{k=1}^{r} C^k \phi(a_k) = \sum_{j=1}^{r} b_{kj}\,\beta_j\,) = \sum_{k=1}^{r} (\sum_{j=1}^{r} b_{jk}\,C^j)\,\beta_k .$$

But ϕ is an isomorphism of algebras $< = >$

$$[\phi(a_i)\ .\ \phi(a_j)] = \sum_{k=1}^{r} C^k\,\phi(a_k) .$$

Therefore $(*)$ and $(**)$ give upon equating coefficients:

ϕ is an isomorphism $< = >$ $H D^k H^T = \sum_{j=1}^{r} b_{jk}\,C^j$, for $k=1, \ldots, r$

This is (8).

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$Z \in \Gamma$ *está anotado en* L^p , $1 < p < \infty$, *si* Γ *es una curva simple cerrada de clase* $C^{1+\epsilon}$, $\epsilon > 0$.

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