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ALGORITHMIC REPRESENTATION OF WERMUS' CONSTRUCTIONS OF ORDINAL NUMBERS

by

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In his paper [3] Wermus defines ordinal numbers as "*Z*-symbols" of the form $[a_1,\cdots,a_k]$, $k \geqslant 1$ and $[a_1]=a_1$, where the a_j, $1 \leqslant j \leqslant k$, are natural numbers or *Z*-symbols. It will be demonstrated that Wermus' constructions can be described by means of a special Neumer algorithm, the constructive algorithm of [1], part I and V resp. a descriptive algorithm of [2], part III ; more precisely : that there can be established a 1-1 correspondence between *Z*-symbols $[a_1,\cdots,a_k]$ and algorithmic symbols $T[a_1,\cdots,a_k]$ such that $[a_1,\cdots,a_k]$ and $T[a_1,\cdots,a_k]$ represent the same ordinal number. This comparision of the two systems further allows the determination of the least ordinal number which is inaccesible by Wermus' constructions in [3].

A few notations and assumtions are needed. A symbol $lim\,A(n)$ is an abreviation for $\lim_{n<\omega} A(n)$. The symbol σ_β, β ordinal number $>\sigma$, denotes the sequence consisting of β zeros ; especially σ_σ is the void sequence. If $a(VJ\beta)^r$ is an algorithmic symbol then let $a(VJ\beta)^\sigma = a$, $a(VJ\beta)^1 = a\,VJ\beta$, $a\,VJ\sigma = a\,V$. Assume for the *Z*-symbol $[a_1, a_2, \ldots, a_{s+1}, a_{s+2}]$, $s > -1$, that

(1.0) $\quad [a_1, a_2, \cdots, a_{s+1}, a_{s+2}] = a_1 < \omega \quad$ if $\quad s = -1$

(1.1) $\sigma \leqslant a_1 < [\sigma, a_2 + 1, \cdots, a_{s+1}, a_{s+2}]$

(1.2) $\sigma \leqslant a_2 < [\sigma_2, a_3 + 1, \cdots, a_{s+1}, a_{s+2}]$

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(1.s+1) $\sigma \leqslant a_{s+1} < [\sigma_{s+1}, a_{s+2} + 1]$

(1.s+2) $\sigma < a_{s+2} < [\sigma_{s+2}, 1]$

and define

(2) $Ta = a$ if the -symbol $a < \omega$;

(3) $T([\sigma, 1] + 1) = (T[\sigma, 1]) + 1$ for the least transfinite Z-symbol $[\sigma, 1]$

Then the following "representation theorem" can be proved :

THEOREM 1.

$T[a_1, a_2, \cdots, a_{s+1}, a_{s+2}] =$

$((\ldots((1(V\ s)^{Ta_{s+2}})(V\ s\text{-}1)^{Ta_{s+1}})\ldots)(V\ 1)^{Ta_3})V^{Ta_2} + Ta_1$

where

$Ta_j = a_j$ if $a_j < \omega$, $1 \leqslant j \leqslant s+2$,

$T[a_1, a_2, \cdots, a_{s+1}, a_{s+2}] = a_1$ if $s = -1$,

$T[a_1, a_2, \cdots, a_{s+1}, a_{s+2}] = 1V^{Ta_2} + Ta_1$ if $s = \sigma$.

Since the system of the Z-symbol is well - ordered (cf. [3], p.3σ3) the proof will be given by transfinite recursion ; it relies heavily on the "fundamental sequences" $X = \{X^{(n)}\}$ which Wermus assigns to each Z-symbol X of the second Kind (cf. [3], p. 311-312) and on the definitions of the "operators" V and (cf. [1], p. 397-398).

PROOF OF THE THEOREM.

$s = -1$ implies $[a_1] = a_1 < \omega$ and $T[a_1] = a_1$ by (2). Let $s \geqslant \sigma$ and $[a_1, a_2, \cdots, a_{s+2}] =$. Assume :

1. $a_1 > \sigma$, a_1 of the first Kind.

Put $a_1 = \delta + 1$ and $= [\sigma, a_2, \cdots, a_{s+2}] + a_1 =$ $+ \delta + 1$. Now $T(\ + \delta) =$

$T + T\delta$ by hypothesis and since $Z' + \delta$ and $TZ + T\delta$ represent the same ordinal number the same holds for $Z' + \delta + 1$ and $TZ' + T\delta + 1$; therefore $T[a_1, a_2, \cdots, a_{s+2}] = T[\sigma, a_2, \cdots, a_{s+2}] + Ta_1$.

2. $a_1 > \sigma$, a_1 of the secons Kind.

Then $[a_1, a_2, \cdots, a_{s+2}] = \{[a_1^{(n)}, a_2, \cdots, a_{s+2}]\}$ where $\mathfrak{J} a_1 = \{a_1^{(n)}\}$. By hypothesis $T[a_1^{(n)}, a_2, \cdots, a_{s+2}] = T[\sigma, a_2, \cdots, a_{s+2}] + Ta_1^{(n)}$ and $[\sigma, a_2, \cdots, a_{s+2}] + a_1^{(n)}$, $n < \omega$, resp. $\lim a_1^{(n)}$ and $\lim Ta_1^{(n)}$ represent the same ordinal numbers ; therefore $\lim T[a_1^{(n)}, a_2, \cdots, a_{s+2}] = T[\sigma, a_2, \cdots, a_{s+2}] + \lim Ta_1^{(n)} = T[\sigma, a_2, \cdots, a_{s+2}] + Ta_1 = T[a_1, a_2, \cdots, a_{s+2}]$. By 1 and 2 the "additive part" of the theorem is proved.

3. $a_1 = \sigma$ and $Z = [\sigma_{s+1}, 1]$, $s \geqslant \sigma$.

a) $s = \sigma$: $\mathfrak{J}[\sigma, 1] = \{n\}$ and by (2) and definition of V $Tn = n$, $\lim Tn = \lim n = 1V^1$ or $T[\sigma, 1] = 1V^1$.

b) $s > 1$: $\mathfrak{J}[\sigma_{s+1}, 1] = \{Z^{(n)}\}$ where $Z^{(1)} = [\sigma_s, 1]$, $Z^{(1+n)} = [\sigma_s, Z^{(n)}]$ for $n \geqslant 1$. The hypothesis yields $TZ^{(1)} = 1(VJ^{s-1})1$, $TZ^{(1+n)} = 1(VJ^{s-1})^{TZ^{(n)}}$ and $\lim TZ^{(n)} = \lim\{1(VJ^{s-1})1, 1(VJ^{s-1})1, 1(VJ^{s-1})1(VJ^{s-1})1, 1(VJ^{s-1})1(VJ^{s-1})1(VJ^{s-1})1, \ldots\} = 1(VJ^{s-1})\ldots = 1(VJ^s)1$. Thus $T[\sigma_{s+1}, 1] = 1(VJ^s)1$.

4. $a_1 = \sigma$ and $\mathfrak{Z} = [\sigma_t, a_{t+1}, \cdots, a_{s+2}]$, $a_{t+1} \neq \sigma$, $1 \leqslant t \leqslant s+1$, $Z \neq [\sigma_{s+1}, 1]$.

a) a_{t+1} of the first Kind.

Put $a_{t+1} = \delta + 1$. $\mathfrak{J}[\sigma_t, \delta+1, \ldots, a_{s+2}] = \{Z^{(n)}\}$ where $Z^{(1)} = [\sigma_t, \delta, \ldots, a_{s+2}]$ $Z^{(1+n)} = [\sigma_{t-1}, Z^{(n)}, \delta, \ldots, a_{s+2}]$, $n \geqslant 1$. If now $t = 1$ then $T\mathfrak{Z}^{(1)} = (\ldots(1(VJ^s)^{Ta_{s+2}})\ldots)V^{T\delta}$ and since obviously $\mathfrak{Z}^{(n)} = \mathfrak{Z}^{(1)}n$, $n \geqslant 1$, $\lim T\mathfrak{Z}^{(n)} = \lim T(\mathfrak{Z}^{(1)}n) = ((\ldots(1(VJ^s)^{Ta_{s+2}})\ldots)V^{\delta}V$.

Therefore $T[\sigma, a_2, \cdots, a_{s+2}] = (\ldots(1(V^{\ s})^{Ta_{s+2}})\ldots)V^{Ta_2}$. If $t>1$

then $TZ^{(1)} = (\ldots(1(VJ^{\ s})^{Ta_{s+2}})\ldots)(VJ^{\ t\text{-}1})^{T\delta}$, $TZ^{(1+n)} = (TZ^{(1)})(VJ^{\ t\text{-}2})^{TZ^{(n)}}$, $n>1$ and $\lim TZ^{(n)} = \lim \{TZ^{(1)}, (TZ^{(1)})(VJ^{\ t\text{-}2})^{TZ^{(1)}}$

$(TZ^{(1)}(VJ^{t\text{-}2})(TZ^{(1)})(VJ^{t\text{-}2})^{TZ^{(1)}}$ $\ldots\} = (TZ^{(1)}(VJ^{t\text{-}2})J = ((\ldots(1(VJ^s)^{Ta_{s+2}})\ldots)(VJ^{t\text{-}1})^{T\delta})VJ^{t\text{-}1}$ which implies that the theorem holds for

$Z = [\sigma_t, a_{t+1}, \cdots, a_{s+2}]$

b) a_{t+1} of the second Kind.

Put $a_{t+1} = \delta$, $\mathfrak{F}\delta = \{\delta^{(n)}\}$. Then $\mathfrak{F}[\sigma_t, \delta, \cdots, a_{s+2}] = \{[\sigma_t, \delta^{(n)}, \ldots, a_{s+2}]\}$. Now $\lim \delta^{(n)} = \delta$ implies $\lim T\delta^{(n)} = T\delta$ and since

$T[\sigma_t, \delta^{(n)}, \ldots, a_{s+2}] = (\ldots(1(VJ^{\ s})^{Ta_{s+2}})\ldots)(VJ^{\ t\text{-}1})^{T\delta^{(n)}}$ by hypothesis also $\lim T[\sigma_t, \delta^{(n)}, \ldots, a_{s+2}] = (\ldots(1(VJ^s)^{Ta_{s+2}})\ldots)(VJ^{\ t\text{-}1})^{T\delta}$.

This completes the proof .

Remark concerning the restrictions (i.i) $,1<i<s+2,$ for Z-symbols. By representation theorem $T[\sigma_k, a_{k+1}, \cdots, a_{s+2}] = Q$ where

$Q = (\ldots(1(VJ^{\ s})^{Ta_{s+2}})\ldots)(VJ^{\ k\text{-}1})^{Ta_{k+1}}$ and

(4) $TZ = Q(VJ^{k\text{-}2})^{Ta_k}$ for $Z = [\sigma_{k\text{-}1}, a_k, a_{k+1}, \cdots, a_{s+2}]$, $2<k<s+2$.

Obviously (4) can be proved by recursion only if $Ta_k < T$.

(5) Suppose now $a_k = [\sigma_k, a_{k+1}+1, \cdots, a_{s+2}]$ (cf.(1.k)). Then Ta_k = $Q(VJ^{k\text{-}1})$ and Ta_k is the first critical number of the sequence $\{Q(VJ^{\ k\text{-}2})\lambda\}$; *i.e.* $Q(VJ^{\ k\text{-}2})^{Ta_k} = Ta_k$ since $Q(VJ^{\ k\text{-}1}) = \lim\{Q(VJ^{\ k\text{-}2}),$

$Q(VJ^{k\text{-}2})^{Q(VJ^{k\text{-}2})}$ $Q(VJ^{k\text{-}2})^{Q(VJ^{k\text{-}2})^{Q(VJ^{k\text{-}2})}}$ $, \ldots\}$ (cf. [2], 1 p. 135 (ln)).

Assuming (5) therefore amounts to $TZ = Ta_k$ and the recursive argumentation in the proof of theorem 1 must fail without (1.i), $1\leqslant i \leqslant s+2$.

Moreover, the constructive algorithm does not make use of symbols.

(6) $Q(V\ J^{t})^{u}$ where $u \geq Q(V\ J^{t+1})$.

But it is clear that once theorem 1 is proved under the restrictive conditions (1.i), it also holds if these assumptions are dropped : one only has to replace the constructive algorithm by a descriptive algorithm ([2], III) which contains symbols of the form (6) and if any of (1.i) do not hold one identifies formally $T[a_1, \cdots, a_i, \cdots, a_{s+2}]$ with $(\ldots((\ldots)(V\ J^{i-2})^{Ta_i})\ldots)V^{Ta_2} + Ta_1$.

This identification will be carried out now and in this way justified first for a very general situation.

Wermus defines "amplification A_k" of a Z-symbol by

(7) $A_k[a_1, \cdots, a_{k-1}, a_k, a_{k+1}, \cdots, a_{s+2}] = [a_1, \cdots, a_{k-1}, [a_k, \sigma_{k-1}, a_{k+1}, \cdots, a_{s+2}]]$, $1 < k < s+2$, where on the right side at least (1.k) does not hold. Now

(7') $A_k[a_1, \cdots, a_k, \cdots, a_{s+2}] = [a_1, \cdots, a_k, \cdots, a_{s+2}]$ (cf. [3], p. 317, th. 91).

(7) and (7') suggest then, that also

(8) $T[a_1, \cdots, a_{k-1}, [a_k, \sigma_{k-1}, a_{k+1}, \cdots, a_{s+2}]] = T[a_1, \cdots, a_{k-1}, a_k, a_{k+1}, \cdots, a_{s+2}]$

should hold. By theorem 1 or by the identification $T[\sigma_k, a_{k+1}, \cdots, a_{s+2}] = Q$, where $Q = (\ldots(1(VJ^{s})^{Ta_{s+2}})\ldots)(VJ^{k-1})^{Ta_{k+1}}$,

(9) $T[a_k, \sigma_{k-1}, a_{k+1}, \cdots, a_{s+2}] = Q + Ta_k$,

(10) $T[\sigma_{k-1}, a_k, a_{k+1}, \cdots, a_{s+2}] = Q(VJ^{k-2})^{Ta_k}$,

(11) $T[\sigma_{k-1}, [a_k, \sigma_{k-1}, a_{k+1}, \cdots, a_{s+2}]] = 1(VJ^{k-2})^{Q+Ta_k}$, using (9).

By the algorithmic "theorem of parallelism" ([2], II p. 44) is $Q(VJ^{k-2})^{\lambda} = 1(VJ^{k-2})^{Q+\lambda}$, λ arbitrary, and therefore with (10), (11).

(12) $T[\sigma_{k-1}, a_k, a_{k+1}, \cdots, a_{s+2}] = T[\sigma_{k-1}, [a_k, \sigma_{k-1}, a_{k+1}, \cdots, a_{s+2}]]$.

A slight generalization of (12) yields (8).

Since now every Z-symbol $[a_1, \cdots, a_i, \cdots a_{s+2}]$ which does or does not fulfill (1.i), $1 < i < s+2$, may be written in the amplified form, then with (8) there is always a corresponding symbol of a descriptive algorithm for it.

In the two remaining cases, where (1.1) or $(1.\ s+2)$ do not hold, identification is trivial. Suppose $a_1 \geqslant [\sigma, a_2+1, \cdots, a_{s+2}]$ resp. $a_{s+2} > [\sigma_{s+2}, 1]$ and let the identification be performed for a_1 resp. a_{s+2} (if necessary). Then in both cases identify $T[a_1, a_2, \cdots, a_{s+2}]$ with

$(\ldots(1(V\ _s)^{Ta_{s+2}})\ldots)^{Ta_2} + Ta_1$.

Conclusión.

THEOREM 2.

Theorem 1 holds without assuming (1.i), $1 \leqslant i \leqslant s+2$, if the algorithmic symbols belong to a descriptive algorithm.

Wermus considers only Z-symbols $Z = [a_1, \cdots, a_t]$ with $t < \omega$. By theorem 1 and its generalization it is obvious that there always exist natural numbers $f < g$ such that $[\sigma_f, 1] \leqslant Z < [\sigma_g, 1]'$. Now $lim[\sigma_n, 1] = lim\ T[\sigma_n, 1] = lim\ 1VJ^n = 1VJ^{1V}$.

Therefore

THEOREM 3.

The least ordinal number which is inaccessible by Wermus' constructions in [3] is $1VJ^{1V}$.

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