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MAXIMAL OPEN COMPACT SUBGROUP OF THE PROJECTIVE SYMPLECTIC GROUP OVER A LOCALLY COMPACT DISCRETE VALUATION FIELD

by

Nelo D. Allan

University of Notre Dame

and

Universidad Nacional de Colombia

For the purpose of describing maximal arithmetic groups one tecnique would be to study the local correspondent problem and then try to put the local results together. The general theory of Hijikata and Beuhat-Tits handles the local case. However in order to globalize one needs to know precise information about the orders and lattice involved. This is an expository note with the objective of providing the information we need to give complete proofs to the results announced [5]. Our goals are to explicitely determine, up to inner automorphisms, all maximal open compact subgroups of the Projective Symplectic Group, $PS_{p_n}(k)$, in the case where k is a locally compact discrete valuation field whose residue class field has characteristic different from 2. Let $S_{p_n}(k)$ denote the Symplectic

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Group over the algebraic closure $\bar{k}$ of k and let $S_{p_n}(o)$ denote its subgroup consisting of all matrices which have entries in the ring of integers o of k. Our problem is then to determine up to inner automorphism all subgroups of $S_{p_n}(k)$ which are commensurable to $S_{p_n}(o)$ and are maximal with this property. Let L_s be a lattice in k^{2n} having s elementary divisors equal to o and $n-s$ equal to the maximal ideal $\mathfrak{p}$ of o, and let us denote by $\Delta(L_s)$ its stabilizer in $S_{p_n}(k)$. It is well known that the normalizer $N(\Delta(L_s))$ of $\Delta(L_s)$ in $S_{p_n}(\bar{k})$, $s = 0, 1, \ldots, [n/2]$, $[n/2]$ being the biggest integer not greater than $n/2$, yield us maximal groups. We shall prove that the normalizers in $S_{p_n}(\bar{k})$ of the intersections $\Delta_s = \Delta(L_s) \cap \Delta(L_{n-s})$ are also maximal for $s = [n/2] + 1, \ldots, n$. Hence we get $n+1$ maximal groups which are not conjugate under inner automorphisms of $S_{p_n}(\bar{k})$; we prove that any other maximal group is conjugate to one of these.

1. Generalities. First we shall introduce our notation and then, for the sake of completeness we shall restate some of the general results on maximality repeating several proofs.

Throughout this paper k will be a locally compact discrete valuation field o its ring of integers, $\mathfrak{p} = (\pi)$ its prime ideal, where π is any generator of $\mathfrak{p}$, and U the group of units of o. Let $\bar{k}$ be the algebraic closure of k; and let $M_n(S)$ denote the ring of all n by n matrices with entries in a subring S of $\bar{k}$. We shall say that a subgroup Δ of a semisimple linear group $G \subset M_n(k)$, defined over k is arithmetic, if it is commensurable to $G_0 = G \cap M_n(0)$, i.e., $\Delta \cap G_0$ has finite index in both Δ and G_0. We say that an arithmetic group Δ is maximal (resp. maximal in $G_k = G \cap M_n(k)$) if it is not a proper subgroup of any other arithmetic group (respectively, contained in G_k). From now on $G = S_{p_n}(\dot{k})$ will be the Symplectic Group, i.e., G is the group of all matrices g in $M_{2n}(\dot{k})$ such that ${}^t gJg = J$ where $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, E being the n by n identity matrix. Let $G' = Gp_n(\dot{k})$ be the similitude group of J, i.e., G' is the set of all $2n$ by $2n$ matrices g such that ${}^t gJg = \mu(g) J$ for some $\mu(g) \in \dot{k}$. The Projective Symplectic Group $PS_{p_n}(\dot{k})$ or simply PS_{p_n}, is to be defined as the quotient $S_{p_n}(\dot{k})$ by its center. We shall consider the following representation of $PS_{p_n}(\dot{k})$ as a linear group. If we map, in a natural way, the group $Sp_n(\dot{k})$ into the group I of all inner automorphisms of $M_{2n}(\dot{k})$, then its image is centerless, and isomorphic to $PSp_n(\mathring{k})$. We know that if we choose a basis for $M_{2n}(\dot{k})$ then the group I has a linear representation in $M_N(k)$, $N = 4n^2$. Hence we obtain a linear representation of PSp_n which is easily seen to be defined over k, and also we have a k-rational mapping from $Sp_n(\dot{k})$ onto $PSp_n(\dot{k})$. This mapping clearly sends the normalizer in G of $Sp_n(0)$ onto $PSp_n(0)$. If Δ is an arithmetic group contained in G_k, then Δ is open compact in G_k, hence by reasons of dimensions, [1] § 33, Δ is linearly dense in the sense that the k-algebra $A(\Delta, k)$ generated by Δ in $M_{2n}(k)$ coincides with $M_{2n}(k)$

because in our case the k-algebra generated by $Sp_n(k)$ is $M_{2n}(k)$. We shall denote by $N(\Delta)$ (respectively by $N_k(\Delta)$) the normalizer of Δ in $Sp_n(\bar{k})$ (respectively in $Sp_n(k)$). It is easy to see that $N_k(\Delta)$ is also open compact and therefore arithmetic. Let us denote by $A(\Delta, 0)$ the 0-order generated by Δ in $M_n(k)$. Let us fix once for all a complete set of representatives of U/U^2 say $\{\varepsilon_1, \dots, \varepsilon_t\}$ which is finite because the residue class field of k has characteristic different from two. For the sake of completeness we shall prove the following lemma :

Lemma 1. 1) If Δ is an arithmetic group such that $\Delta \subset Sp_n(k)$ and if $g \in N(\Delta)$, $g \notin \mathbf{G}_k$, then $g = g'/\sqrt{\alpha}$, where $\alpha \in 0$, $g' \in M_{2n}(k)$, i.e., $g' \in \mathbf{G}p_n(k)$ and $\mu(g') = \alpha$.

2) If Δ is an arithmetic group $Sp_n(\bar{k})$ and $\Delta = \Delta' \cap Sp_n(k)$, then $\Delta' \subset N(\Delta)$. In particular if Δ' is maximal, then $N_k(\Delta) = \Delta$ and $N(\Delta)/\Delta$ is abelian.

Proof. If $g \in N(\Delta)$, then g normalizes $A(\Delta, k) = M_{2n}(k)$, hence $g^{-1} e_{ij} g \in M_{2n}(k)$ for all $i, j = 1, \dots, 2n$, and as $g^{-1} = -J\,{}^t g J$ we get that $g_{ij}\, g_{sm} \in k$ for all $i, j, s, m = 1, \dots, 2n$, hence $g_{ij}^2 \in k$ and if $g_{11} \neq 0$, then $g_{ij} = \lambda_{ij} g_{11}$, $\alpha' = g_{11}^2 \in k$, $\lambda_{ij} \in k$, and consequently $\alpha' = \lambda^2 \alpha$ with $\alpha \in 0$, $\lambda \in k$. Hence assertion 1. Next if $g \in \Delta'$, then $g \in N(\Delta_o)$ where Δ_o is the kernel of representation of Δ' as group of permutations of Δ'/Δ. Clearly Δ_o in arithmetic. Hence $g = g'/\sqrt{\alpha}$ and $g^{-1} \Delta' g = (g')^{-1} \Delta (g') \subset Sp_n(k)$ $\Delta' = \Delta$. From the assertion 1, it follows that every element of $N(\Delta)/\Delta$ has order at most two, therefore this group is an abelian group of order at most 2^{t+1} where t is the order of U/U^2. Hence as $[N(\Delta):\Delta]=[N(\Delta):N_k(\Delta)][N_k(\Delta):\Delta]$ we have that $N(\Delta)$ is arithmetic. If Δ' is maximal, then $\Delta' \subset N(\Delta)$, $\Delta' = N(\Delta)$, hence $N_k(\Delta) = N(\Delta) \cap Sp_n(k) = \Delta$.

q. e. d.

For the sake of simplicity we shall denote by g_α the element of $N(\Delta)$ which can be written as $g'/\sqrt{\alpha}$. Clearly given α if it exists such $g_\alpha \varepsilon N(\Delta)$, then it is unique, modulo Δ, and α is uniquely determined modulo o/o^2. We shall denote by $U(\Delta)$ the set of all $g_\alpha \varepsilon N(\Delta)$ such that α can be chosen in U. Hence $U(\Delta)/\Delta$ is injected in U/U^2 and $N(\Delta)/U(\Delta)$ is a group of order at most 2; it has order 2 precisely when there exists an element $g_\pi \varepsilon N(\Delta)$ for some generator π of $\mathfrak{p}$. Clearly $U(\Delta)$ is generated by Δ and g_α where α runs in a subset of $\{\varepsilon_1, \dots, \varepsilon_t\}$.

Lemma 2. Let Δ and Δ_1 be two arithmetic groups contained in $Sp_n(k)$ such that $N_k(\Delta) = \Delta$ and $N_k(\Delta_1) = \Delta_1$. If $N(\Delta)$ and $N(\Delta_1)$ are conjugate in $Sp_n(\bar{k})$, then Δ is conjugate to Δ_1 in $Gp_n(k)$.

Proof. Let $g^{-1}N(\Delta)\,g = N(\Delta_1)$. We set $\Delta_2 = g\,\Delta_1\,g^{-1} \cap \Delta$ and $\Delta_o = \Delta_2 \cap \Delta_1$. Hence Δ_o and Δ_2 are arithmetic and $g^{-1}\Delta_o g \subset \Delta_1$ because inner automorphisms preserve indices of subgroups. As Δ_o and Δ_1 are linearly dense we get that $g^{-1}M_n(k)\,g = M_n(k)$, hence

$$g^{-1}\Delta\, g = g^{-1}N(\Delta)\, g \cap Sp_n(k) = N(\Delta_1) \cap Sp_n(k) = \Delta_1 .$$

Finally as $g \in N(Sp_n(k))$, then $g = g' / \sqrt{\alpha}$, $g' \in Gp_n(k)$ and $\mu(g') = \alpha$.

q. e. d.

It is clear that there exists a one to one correspondence between the set of all maximal arithmetic groups in Sp_n and the set of all maximal arithmetic groups in PSp_n. $Sp_n(k)$ acts on $V = k^{2n}$ and if L is a lattice in V, then we can always find a basis of L in such a way that $L = 0e_1 + \ldots + 0e_n + A_1 e_{n+1} + \ldots + A_n e_{2n}$ (see [7], p. 35) where $ord_p A_1 \leqq \ldots \leqq ord_p A_n$, $Ord_p A = a$ meaning that $A = p^a$, and if we set $f(x, y) = {}^t x J y$, then $f(e_j, e_{n+j}) = 1 = -f(e_{n+j}, e_j)$ and $f(e_i, e_j) = 0$ otherwise. We call this basis a canonical basis for L; we set $n(L) = A_1$ and call $\{A_i / A_1, \ i = 1, \ldots, n$ the elementary divisors of L; it is well known that these ideals are invariants of L. Let $\mathbf{L}$ be an order in $M_n(k)$; we set $\mathbf{L}^\sigma = J\,{}^t\mathbf{L}J$, and we say that $\mathbf{L}$ is σ-invariant if $\mathbf{L}^\sigma = \mathbf{L}$. Clearly if L is a lattice and $End\, L = \{ g \in M_n(k) \mid gL \subset L \}$, then $End_\sigma L = (End\, L) \quad (End\, L)^\sigma$ is σ-invariant, and any σ-invariant order $\mathbf{L}$ is contained in some $End_\sigma L$. Let $\mathbf{L}$ be an order and let $(\mathbf{L})_{ij}$, or simply L_{ij}, be the ideal generated by all (i, j)-entries of all $g \in \mathbf{L}$. We let $[A_i : A_j] = (A_i / A_j) \quad 0$. We say that $\mathbf{L}$ is a direct summand if $\mathbf{L}$ is the direct sum as 0-module of $L_{ij} e_{ij}$; this is true if $e_{ii} \in \mathbf{L}$ for all $i = 1, \ldots, 2n$.

Lemma 3. (Hijikata). We set $\mathbf{L} = A(\Delta(L), 0)$, then $\mathbf{L}$ is a direct summand, $\mathbf{L} = End_\sigma \mathbf{L}$ and for all $i, j = 1, \ldots, n,$

$$\mathrm{L}_{ij} = \mathrm{L}_{n+j\ n+i} = \mathrm{L}_{in+j}\mathrm{A}_j = \mathrm{L}_{n+j\,j}\mathrm{A}_i^{-1} = [\,\mathrm{A}_j : \mathrm{A}_i\,].$$

Moreover $End_\sigma L$ is maximal σ - invariant if and only if the elementary divisors of L are square free.

Proof. It follows from [4], theorem 1, and also [6], theorem 5.4. For the sake of completeness we shall compute this order. Clearly $L \subset End_\sigma L$. We first observe that if $g(H) = \begin{pmatrix} E & H \\ 0 & E \end{pmatrix}$, $H = ae_{jj}$, $a \,\varepsilon\, \mathrm{A}_j^{-1}$, then $g(H) \,\varepsilon\, \Delta(L)$ and $a\,e_{jn+j} \,\varepsilon\, \mathbf{L}$; similarly ${}^t g(H) \,\varepsilon\, \Delta(L)$, $ae_{n+j\,j} \,\varepsilon\, \mathbf{L}$, if $H = ae_{jj}$, $a \,\varepsilon\, \mathrm{A}_j$, for all $j = 1, \ldots, n$; hence for these indices $\mathrm{L}_{n+j\,j} = \mathrm{A}_j$, $\mathrm{L}_{j\,n+j} = \mathrm{A}_j^{-1}$ and consequently $\mathbf{e}_{ii} \,\varepsilon\, \mathbf{L}$ for all $i = 1, \ldots, 2n,$ and $j = 1, \ldots, n$. Hence both $\mathbf{L}$ and $End_\sigma L$ are direct summands. Now if $g(A, D) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, $A = E + ae_{ij}$, $i \neq j$, ${}^t DA = E$, then $g \,\varepsilon\, \Delta(L)$ if and only if $gL = L$, or equivalently $A(0\mathbf{e}_1 + \ldots + 0\mathbf{e}_n) \subset 0\mathbf{e}_1 + \ldots + 0\mathbf{e}_n$ and $D(\mathrm{A}_1\mathbf{e}_1 + \ldots + \mathrm{A}_n\mathbf{e}_n) \subset \mathrm{A}_1\mathbf{e}_1 + \ldots + \mathrm{A}_n\mathbf{e}_n$, and this is valid if and only if $a \,\varepsilon\, 0$ and $a\mathrm{A}_i \subset \mathrm{A}_j$, i.e., $a \,\varepsilon\, [\mathrm{A}_j : \mathrm{A}_i]$. Consequently $\mathrm{L}_{ij} = [\,\mathrm{A}_j : \mathrm{A}_i\,]$. Since both $\mathbf{L}$ and $End_\sigma L$ are σ - invariant we get that $\mathrm{L}_{n+jn+i} = \mathrm{L}_{ij}$, $\mathrm{L}_{n+ij} = \mathrm{L}_{n+ji}$ and $\mathrm{L}_{in+j} = \mathrm{L}_{jn+i}$. Now as $\mathbf{L}$ is a direct summand order the other assertions follows from $\mathrm{L}_{ij}\mathrm{L}_{jn+j} = \mathrm{L}_{in+j}$ and $\mathrm{L}_{n+in+j}\mathrm{L}_{n+jj} = \mathrm{L}_{n+ij}$.

q. e. d.

2. Necessary conditions. Our objectives now are to study necessary conditions for the maximality of the normalizer $N(\Delta)$ of a subgroup Δ of $Sp_n(k)$. We want to find conditions under which a group $\Delta = N(\Delta) \cap Sp_n(k)$, non maximal in $Sp_n(k)$, have its normalizer maximal. We start studying the behavior of the group $U(\Delta)$ with respect to the maximal groups in $Sp_n(k)$ which contains Δ.

Lemma 4. Let $\Delta \subset Sp_n(k)$ be such that $N_k(\Delta) = \Delta$. Let L^* be the union of all $A(\Delta, 0)\, b_\alpha$, for all $b_\alpha = \sqrt{\alpha}\, g_\alpha$, $g_\alpha \varepsilon N(\Delta)$. Then L^* is a σ-invariant 0-order. Moreover if $\alpha \varepsilon \mathrm{U}$, then $b^{-1}_\alpha \varepsilon L^*$.

Proof. Clearly $S = \bigcup A(\Delta, 0)\, b_\alpha$, $\alpha \varepsilon \mathrm{U}$ is an order, because $b_\alpha b_\beta = \lambda b_\lambda \varepsilon S$. Next $L^* = S \cup S b_\pi$ and to prove that L^* is an order, it suffices to show that $b_\pi^2 \varepsilon L^*$, but $b_\pi^2 = \pi\, g_\pi^2 \varepsilon \pi \Delta \subset S$. From $J^t b_\alpha J = \sqrt{\alpha} J^t g_\alpha J = - \sqrt{\alpha}\, g_\alpha^{-1}$ and $\theta = g_\alpha^2 \varepsilon \Delta$, we can write $g_\alpha^{-1} = \theta^{-1} b_\alpha / \sqrt{\alpha}$ hence $\alpha g_\alpha^{-1} = \theta^{-1} b_\alpha$, i.e., L^* is σ-invariant. Finally as $\alpha b_\alpha^{-1} = \theta^{-1} b_\alpha$ and as $\alpha \varepsilon \mathrm{U}$ we get that $b_\alpha^{-1} \varepsilon L^*$.

q. e. d.

Let now K be the field generated by k and all $\sqrt{\varepsilon_j}$ for all $j = 1, \ldots, t$.

Lemma 5. Let $\Delta \subset Sp_n(k)$ be such that $N(\Delta) \subset Sp_n(k)$, i.e., $N(\Delta) = U(\Delta)$. Then there exists a lattice L such that $N(\Delta)$ is contained in $N(\Delta(L))$.

Proof. The order L^*, of lemma 4 is contained in $End_\sigma L$ for sone lattice L with elementary divisors square free. Consequently $\Delta \subset \Delta(L)$ and from the fact that for all $g_\varepsilon \in N(\Delta)$, $(h_\varepsilon)^{-1} \in End_\sigma L$ and $g_\varepsilon^{-1}(End_\sigma L)\, g_\varepsilon = (h_\varepsilon)^{-1}(End_\sigma L)\, h_\varepsilon = End_\sigma L$, we get that g_ε normalizes $End_\sigma L$ and G_k, hence $g_\varepsilon \in N(\Delta(L))$ and $N(\Delta) \subset N(\Delta(L))$.

q. e. d.

Lemma 6. Let Δ be an arithmetic group in $Sp_n(k)$ such that Δ is not maximal in $Sp_n(k)$, $N_k(\Delta) = \Delta$, and $N(\Delta)$ is maximal. Then there exists lattices L' and L'' with same square free elementary divisors such that $\Delta = \Delta(L')\ \Delta(L'')$. Moreover there exists $h \in Gp_n(k)$ such that $L'' = hL'$ and $h \in End_\sigma(L')$ with $\mu(h) = \pi$ and $(h^{-1})\pi$, h^2/π, all lying in $End_\sigma(L')$ for some convenient generator π of $\mathfrak{p}$

Proof. By lemma 2, if $N(\Delta) = U(\Delta)$, then $N(\Delta)$ is not maximal. Hence $N(\Delta) \neq U(\Delta)$, i.e., there exists a g_π, for a conveniently chosen generator π of $\mathfrak{p}$, $g_\pi \in N(\Delta)$; let $L' = End_\sigma(L)$

be a maximal σ-invariant order containing $\mathbf{L}^*$ of lemma 4. By lemma 3, the elementary divisors of L' are square free. Now if $g \in \boldsymbol{U}(\Delta)$, then by the same arguments as in the end of lemma 5, $g \in N(\Delta(L))$. Let now $L'' = h_\pi L'$; hence $\mathbf{L}'' = End_\sigma(L'') = h_\pi(L')h_\pi^{-1}$ and $\Delta(L'') = h_\pi \Delta(L')h_\pi^{-1} = g_\pi \Delta(L')g_\pi^{-1}$. If we set $\Delta' = \Delta(L') \cap \Delta(L'')$, then $\Delta \subset \mathbf{L}^* \cap Sp_n(k) = \Delta(L')$ and as $g_\pi \in N(\Delta)$, $\Delta \subset \Delta(L'')$; hence $\Delta \subset \Delta'$. If $g_\varepsilon \in \boldsymbol{U}(\Delta)$, then h_ε normalizes $\mathbf{L}'$ and modulo $\mathbf{L}'$ commutes with g_π; consequently $g_\varepsilon \in N(\Delta(L''))$ or $\boldsymbol{U}(\Delta) \subset N(\Delta(L')) \cap N(\Delta(L'')) \subset N(\Delta')$; also as $g_\pi^2 \in \Delta$ and the inner automorphism of $M_n(k)$ induced by $(g_\pi)^{-1}$ transforms $\Delta(L')$ onto $\Delta(L'')$ we get that $g_\pi \in N(\Delta')$. Therefore $N(\Delta) \subset N(\Delta')$ and by the maximality of $N(\Delta)$, $N(\Delta) = N(\Delta')$. Finally $\Delta' \subset \mathbf{G}_k \cap N(\Delta) = \Delta$, i.e., $\Delta' = \Delta$. It is clear that if we set $h = h_\pi$, then $\mu(h) = \pi$ and as $h \in \mathbf{L}^*$ as well as $h^2/\pi = g_\pi^2$ and $\pi h^{-1} = h g_\pi^{-2}$ we get that all these lie in $End_\sigma(L')$.

q. e. d.

Let now $f \in \mathbf{G}p_n(k)$, from $\Delta = \Delta(L') \cap \Delta(hL')$ we get that $f\Delta f^{-1} = \Delta(fL) \cap \Delta(h' fL)$ where $h' = f h f^{-1}$. It is easy to see that $\mu(h') = \pi$ and that h', $(h')^2/\pi$, $\pi(h')^{-1} \in End_\sigma(fL')$. Since $N(\Delta)$ is maximal if and only if $N(f\Delta f^{-1})$ is maximal, we may replace Δ by $f\Delta f^{-1}$ and we may assume that $L = L_s$, $s = 0, 1, \ldots, n$, where

L_s is the lattice which in a canonical basis has $A_1 = \ldots = A_s = 0$ and $A_{s+1} = \ldots = A_n = \mathfrak{p}$. We set $\mathsf{L}_s = End_\sigma(L_s)$ and let $\Gamma(L_s)$ denote the $G\mathfrak{p}_n$ units of L_s.

Lemma 7. Let $b \in \mathsf{L}_s \cap \mathsf{G}_k$ be such that $\mu(b) = \pi$, and $b^{-1}\pi$, $b^2/\pi \in \mathsf{L}_s$. Then modulo the operations of $\Gamma(L_s)$ the lattice bL_s can be written : $bL_s = \Sigma\, \mathfrak{b}_i e_i$, where $\mathfrak{b}_i = \mathfrak{p}$ if $1 \leq i \leq p$, or $n+1 \leq i \leq n+s$, or $n+s+q+1 \leq i \leq 2n$; $\mathfrak{b}_i = \mathfrak{o}$ if $p+1 \leq i \leq n$, and $\mathfrak{b}_i = \mathfrak{p}^2$ otherwise, i. e., $n+s+1 \leq i \leq n+s+q$, where $p+q = n-s$.

Proof. First of all from πb^{-1}, $b \in \mathsf{L}_s$ it follows that $b^{-1}\pi L_s$, $bL_s \subset L_s$; hence $\mathfrak{p}L_s \subset bL_s \subset L_s$. We claim that we can find a basis for V which is a canonical basis for L_s yielding a basis for bL_s as an $\mathfrak{o}$-module. We shall construct this basis by induction. Let us assume that our assertion is true for dimensions smaller than 2n. Let (e_i), $i = 1, \ldots, 2n$ be a canonical basis for L_s. If for all $x \in bL_s$, $x = x_1e_1 + \ldots + x_{2n}e_{2n}$, both $x_1 \in \mathfrak{p}$ and $x_{n+1} \in \mathfrak{p}A_1$, then letting $U = \langle e_1, e_{n+1}\rangle^{\perp}$ be the orthogonal complement of the span of e_1 and e_{n+1}, we have $bL_s = \mathfrak{p}\, e_1 + \mathfrak{p}A_1 e_{n+1} + (bL_s \cap U)$. It suffices to apply induction to $bL_s \cap U$ and we are done. If there exists $x \in bL_s$ such that $x_1 \notin \mathfrak{p}$, then we may assume $x_1 = 1$ and by taking $e'_1 = x$ and $U = \langle e'_1, e_{n+1}\rangle^{\perp}$ we get that $f(e'_1, e_{n+1}) = 1$ and $L_s = \mathfrak{o}e'_1 + A_1 e_{n+1} + (L_s \cap U)$.

If $x \in h L_s$, $x = x_1 e'_1 + \dots$, we write $y = x - x_1 e'_1 = x_{n+1} e_{n+1} + x_o$, $x_o \in U$. Since $f(y, e'_1) = - x_{n+1}$, $n(hL_s) = n(h) n(L_s) = pA_1$, it follows from $e'_1 \in hL_s$ and $pA_1 e_{n+1} \in hL_s$ that $hL_s = 0 e'_1 + pA_1 e_{n+1} + (hL_s \cap U)$. Now we apply the induction hypothesis to $L_s \cap U$ and we are done. In the case where for all $x \in hL_s$, $x_1 \in p$, but there exists x such that $x_{n+1} \notin pA_1$, we set $e'_{n+1} = x$ and apply similar argument. Also similar argument completes the induction in the case where $n = 1$. Therefore we can find a canonical basis for L_s such that in this basis $hL_s = B_1 e_1 + \dots + B_{2n} e_{2n}$. It is not difficult to see that we may always replace L by gL, $g \in Gp_n(k)$ in such way that $A_1 = 0$, i.e., $s > 0$. Since $pL_s \subset hL_s \subset L_s$ we have $pA_j \subset B_j \subset A_j$; hence B_j is either p or p^2 if $n + s < j < 2n$ and B_j is either 0 or p, if $1 < j \leqq n+s$. Now if $hL_s = p e_i + 0 e_{n+i} + L$, for some $1 \leqq i \leqq s$, then we can replace h by gh, $g \in \Gamma(L)$ such that $ghL_s = 0 e_i + p e_{n+i} + L'$, i.e., g is the operation which interchanges e_i and $-e_{n+i}$, $i \leqq s$. Hence we may assume that $ord_p B_i \geqq ord_p B_{n+i}$, $1 < j \leqq s$. Similarly if $B_i = p$ and $B_{n+i} = p$ $s < i \leqq n$, then we may replace h modulo $\Delta(L_s)$ in such that $B_i = 0$ and $B_{n+i} = p^2$. Using the fact that $n(hL_s) = pA_1$ we see that we cannot have $B_i = B_{n+i} = 0$ for some i, $1 \leqq i \leqq n$; hence we have that $B_{n+i} = p$ if, $1 \leqq i \leqq s$. Remains the case where $B_i = \mathfrak{p}$ and $B_{n+i} = \mathfrak{p}^2$. Now we observe that we can permute $\{e_i, e_{n+i}\}$ by $\{e_j, e_{n+j}\}$ by means of operations of $\Delta(L_s)$, provided that either $1 \leq i, j \leq s$ or $s < i, j \leq n$. Next we put together, by interchanging pairs if necessary, all basis elements $\{\pi e_j\}$, $j \leq s$, if there exists any in such way that $B_j = \mathfrak{p}$, $p \leq s$ and we denote by p their number; we have necessarily $B_{n+j} = \mathfrak{p}$, $j \leq p$. We do the same with the indices $j > s$ such that there exists a generator $\pi^2 e_{n+j}$ and we call q their number and necessarily $B_j = 0$. Also $B_j = 0$, $B_{n+j} = \mathfrak{p}$, $p < j \leq s$. Now let us compute the elementary divisors of hL_s. We change basis by replacing e_i by πe_i and e_{n+i} by $\pi^{-1} e_{n+i}$, whenever $B_i = \mathfrak{p}$. After interchanging pairs of vectors, if necessary, we have a canonical basis where the ideals A_i are either $\mathfrak{p}$, $\mathfrak{p}^2$, or $\mathfrak{p}^3$, according to whether, say, $i \leq u$, $u+1 \leq i \leq t$, and $t+1 \leq i \leq n$, and the elementary divisors are 0, $\mathfrak{p}$, $\mathfrak{p}^2$. As the elementary divisors of L_s and hL_s are the same, we must have $t = n$, $u = s$. Also the indices i such that the corresponding $A_j = \mathfrak{p}^2$ are precisely the ones $i \leq i \leq p$, and $s+1 \leq i \leq s+q$. Therefore $p + q = n - s$.

q. e. d.

3. **Non maximality**. We shall now study the orders in $M_n(k)$ which are generated by the stabilizers of the lattices L_s and hL_s of lemma

7 ; we shall look at the intersection $\Delta(L_s) \cap \Delta(hL_s)$ and prove that whenever $0 \leq p < s,\ n\text{-}s$, S is contained in a bigger order S' which intersect $Sp_n(k)$ in a group Γ such that $N(\Gamma)$ contains $N(\Delta(L_s) \cap \Delta(hL_s))$ properly.

Let L be an order in $M_n(k)$. If $(L)_{ij} = A_{ij}$ we shall set

$$L = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

We observe that if M is another order , $(M)_{ij} = B_{ij}$, then $(L \cap M)_{ij} \subset A_{ij} \cap B_{ij}$. If they are direct summand, then we have the equality. Now with the block notation, we have

$$L_s = \begin{array}{c} \begin{array}{cccc} \overbrace{}^{s} & \overbrace{}^{n-s} & \overbrace{}^{s} & \overbrace{}^{n-s} \end{array} \\ \begin{pmatrix} 0 & p & 0 & 0 \\ 0 & 0 & 0 & p^{-1} \\ 0 & p & 0 & 0 \\ p & p & p & 0 \end{pmatrix} \end{array} \begin{array}{l} \} \ s \\ \} \ n-s \\ \} \ s \\ \} \ n-s \end{array}$$

From now on we fix $h \in \mathbf{L}_s \cap \mathbf{G}p_n(k)$ satisfiying the conditions of lemma 6 and fix a canonical basis for L_s in such that hL_s is written as above. Now we subdivide each matrix $g \in \mathbf{L}_s$ in 64 blocks $g = (g_{ij})$, $i, j = 1, \ldots, 8$, in such way that g_{ij} is either p by p, $s-p$ by $s-p$, or q by q according to whether $i = 1, 4, 5, 8$, or $i = 2, 6$, or $i = 3, 7$. Let $g(p)$ be the matrix (g_{ij}) where

$$g_{18} = g_{45} = E_p, \; g_{26} = E_{s-p}, \; g_{37} = \pi^{-1} E_p$$

$$g_{81} = g_{54} = -\pi E_p, \; g_{62} = -\pi E_{s-p}, \; g_{73} = -\pi^2 E_q$$

and $g_{ij} = 0$ otherwise, i. e.,

$$g(p) = \begin{array}{c} \begin{array}{cccccccc} p & s-p & q & p & p & s-p & q & p \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_p \\ 0 & 0 & 0 & 0 & 0 & E_{s-p} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi^{-1}E_q & 0 \\ 0 & 0 & 0 & 0 & E_p & 0 & 0 & 0 \\ 0 & 0 & 0 & -\pi E_p & 0 & 0 & 0 & 0 \\ 0 & -\pi E_{s-p} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\pi E_q & 0 & 0 & 0 & 0 & 0 \\ -\pi E_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \} \; p \\ \} \; s-p \\ \} \; q \\ \} \; p \\ \} \; p \\ \} \; s-p \\ \} \; q \\ \} \; p \end{array}$$

Clearly $g(p) \varepsilon L_s \cap G p_n$, $g(p)^2 = -\pi E_{2n}$, $\mu(g(p)) = \pi$, and $g(p) L_s = h L_s$. We set $\Delta(p) = \Delta(L_s) \cap \Delta(g(p) L_s)$.

Lemma 8. If $p \neq 0$, $s, n-s$, then $N(\Delta(p))$ is not maximal in G.

Proof. We set $\Delta' = \Delta(L_p) \cap \Delta(g(p) L_p)$; hence $\Delta' = (L_p \cap g(p) L_p g(p)^{-1})$ G_k. As $\Delta(p) = (L_s \cap End_\sigma(g(p)L_s))$ G_k, to prove that $\Delta' \supset \Delta(p)$, properly it suffices to prove that $L_p \cap End_\sigma(g(p) L_p)$ contains $L_s \cap End_\sigma(g(p)L_s)$ properly. If we observe that $End_\sigma(gL) = g\, End_\sigma(L)\, g^{-1}$ and apply this to our cases we see that, after writting these orders in matrix blocks corresponding to the 64 blocks that we subdivide L_s, a direct calculation yields the generator of $L_s \cap End_\sigma(g(p) L_s)$ as follows :

a) πe_{ij} for all (i, j) in the blocks (I, J) where either

$I = 7,\ J = 1, 5, 6, 8$; $I = 1, 2, 4, 5,\ J = 3$

$I = 6, 8,\ J = 1, 2, 4, 5$; $I = 1, 5,\ J = 2, 4$

$(I, J) = (2, 4)$ or $(8, 6)$

b) $\pi^{-1} e_{ij}$ for all (i, j) in the blocks $(I, J) = (6, 3)$, $(7, 2)$, $(7, 3)$, $(7, 4)$ and $(8, 2)$

d) e_{ij} otherwise

Now $L_p \cap End_\sigma(g(p) L_p)$ is the order generated by the above order and

πe_{ij} for all (i, j) in the blocks $(6, 3)$ and $(7, 2)$. Writting in block notation, we have

$$L_s \cap (g(p) L_s g(p)^{-1}) = \begin{array}{c} \begin{array}{cccccccc} p & s-p & q & p & p & s-p & q & p \end{array} \\ \left(\begin{array}{cccccccc} 0 & p & p & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p^{-1} & p^{-1} \\ 0 & 0 & p & 0 & 0 & 0 & p^{-1} & p^{-1} \\ 0 & p & p & p & 0 & 0 & 0 & 0 \\ p & p & p^2 & p & p & 0 & 0 & 0 \\ p & p^2 & p^2 & p^2 & p & p & 0 & p \\ p & p & p^2 & p & p & p & 0 & 0 \end{array}\right) \begin{array}{l} \}\,p \\ \}\,s-p \\ \}\,q \\ \}\,p \\ \}\,p \\ \}\,s-p \\ \}\,q \\ \}\,p \end{array} \end{array}$$

and

$$L_p \cap (g(p) L_p g(p)^{-1}) = \begin{array}{c} \begin{array}{cccccccc} p & s-p & q & p & p & s-p & q & p \end{array} \\ \left(\begin{array}{cccccccc} 0 & p & p & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & p & 0 & 0 & p^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & p^{-1} & p^{-1} & p^{-1} \\ 0 & 0 & p & 0 & 0 & 0 & p^{-1} & p^{-1} \\ 0 & p & p & p & 0 & 0 & 0 & 0 \\ p & p & p & p & p & 0 & 0 & 0 \\ p & p & p^2 & p^2 & p & p & 0 & p \\ p & p & p^2 & p & p & p & 0 & 0 \end{array}\right) \begin{array}{l} \}\,p \\ \}\,s-p \\ \}\,q \\ \}\,p \\ \}\,p \\ \}\,s-p \\ \}\,q \\ \}\,p \end{array} \end{array}$$

Hence if $g = g(H)$, $H = \pi^{-1}(e_{ij} + e_{ji})$, $s < i, j \leqq s+q$, then $g \in \Delta'$, $g \notin \Delta(p)$. Now for all $\varepsilon \in U$ we have g_{ε} = diagonal $\{\eta E, \eta^{-1} E\}$ lies in $N(\Delta) \cap N(\Delta(p))$, $\eta = \sqrt{\varepsilon}$, because ηg_{ε} lies in all direct summand orders ; as h^2/π, $h \in L_s \cap g(p) L_s g(p)^{-1}$ it follows that g_{π} normalizes $L_p \cap g(p) L_p g(p)^{-1} \cap G_k = \Delta'$, i.e., $N(\Delta(p))$ is a proper subgroup of $N(\Delta')$.

q. e. d.

Corollary. If $s \neq 0, n$, then $N(\Delta(0))$ is not maximal.

Proof. It suffices in the above proof omit the blocks in the rows and columns 1, 4, 5 and 8 in all orders.

q. e. d.

Before going on we would like to point out that in the case where $p = s, n-s$, the two orders above coincide. A simple verification shows that if $n - s < s$, then $End_{\sigma}(g(n-s) L_s) = End_{\sigma}(L_{n-s})$; hence.

Lemma 9. (Hijikata). $\Delta(L_s)$ and $\Delta(L_{n-s})$ are conjugated in G, and consequently their normalizer are also conjugate .

4. Maximal groups. We shall discuss now the remaining case, i. e., the case where $hL_s = L_{n-s}$; we shall prove that we get maximal groups in this case. We change notation and denote $\Delta(n-s) = \Delta(L_s) \cap \Delta(L_{n-s})$ by Δ_s ; our objec-

tive is to calculate all lattices left invariant by Δ_s and consequently all maximal subgroups of $Sp_n(k)$ which containis Δ_s; it turns out that they are $\Delta(L_s)$ and $\Delta(L_{n-s})$. We first observe that lemma 9 implies that there is no loss of generality in assuming that $n\text{-}s < s$.

Theorem 1. The normalizer of $\Delta_s = \Delta(L_s) \cap \Delta(L_{n-s})$, $s = [n/2]+1, \ldots, 2n$, is maximal in $Sp_n(\bar{k})$ in the sense that no other subgroup of $Sp_n(\bar{k})$ contains it properly as a subgroup of finite index.

Proof. If $n-s = s$, then as $\Delta(L_s) = \Delta(L_{n-s})$ is maximal in G_k, $N(\Delta(L_s)) = N(\Delta_s)$ is maximal. Hence we may assume that $p = n-s < s$. Since $g_\pi^{-1}\Delta(L_s)\, g_\pi = \Delta(L_{n-s})$, it suffices to prove that Δ_s is contained in precisely two maximal group in $Sp_n(k)$, namely $\Delta(L_s)$ and $\Delta(L_p)$, because if Δ' is maximal and $\Delta' \supset N(\Delta_s)$, then $\Delta = \Delta' \cap G_k$ contains Δ_s and by theorem 1., it is either maximal in G_k or the intersection of two maximal groups, i. e., either $\Delta = \Delta(L_s)$, or $\Delta = \Delta(L_p)$, or $\Delta = \Delta(L_s) \cap \Delta(L_p) = \Delta_s$. Since by lemma 1, $\Delta' = N(\Delta)$, and as $g_\pi \in \Delta'$, and $g_\pi \in N(\Delta(L_s))$, $N(\Delta(L_p))$ we have $\Delta = \Delta_s$ and $\Delta' = N(\Delta_s)$. Let us prove now that Δ_s is contained in precisely two maximal groups in $Sp_n(k)$. First of al Δ_s being contained in both $\Delta(L_s)$ and $\Delta(L_p)$ implies that $L' = A(\Delta_s, 0)$ is contained in $L_s \cap L_p = S$. We shall subdivide every matrix $g \in L$ and every matrix of $L_s \cap L_p$ in 36 blocks in such way that if $g = (g_{ij})$ $i,j = 1, \ldots, 6,$ then g_{ii} is p by p for $i = 1, 3, 4$ and 6, and it is $s\text{-}p$ by $s\text{-}p$ if $i = 2, 5.$ $L_s \cap L_p$ is generated by :

a) πe_{ij} for all (i, j) in the blocks (I, J) where either

$$I = 5, \ J = 1, 2, 3, 4 \ : \ J = 3, \ \ I = 1, 2, 4$$

$$I = 6, \ J = 1, 2, 3, 4, 5 \quad \text{or} \quad J = 2, \ \ I = 1, 4$$

b) $\pi^{-1} e_{ij}$ for all (i, j) in the block (3, 6)

c) e_{ij} otherwise i. e., in block notation :

$$S = L_s \cap L_p = \begin{array}{cc} \begin{array}{cccccc} \overbrace{}^{p} & \overbrace{}^{s\text{-}p} & \overbrace{}^{p} & \overbrace{}^{p} & \overbrace{}^{s\text{-}p} & \overbrace{}^{p} \end{array} & \\ \begin{pmatrix} 0 & p & p & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p^{-1} \\ 0 & p & p & 0 & 0 & 0 \\ p & p & p & p & 0 & 0 \\ p & p & p & p & p & 0 \end{pmatrix} & \begin{array}{l} \} \ p \\ \} \ s\text{-}p \\ \} \ p \\ \} \ p \\ \} \ s\text{-}p \\ \} \ p \end{array} \end{array}$$

Let us prove first that $S_{ij} = L_{ij}$ for all $i, j = 1, \ldots, 2n.$ It is clear that $S \cap Sp_n(k) = \Delta_s$; hence $L_{ij} \subset S_{ij}$. We consider first the elements of Δ_s of the form $g(H)$, $H = a e_{ij}$, $a \in S_{in+i}$, hence $a\,e_{i\,n+i} \in L$ and $L_{i\,n+i} = S_{i\,n+i}$. Similar argument applied to ${}^t g(H)$, $H = b\,e_{ii}$, $b \in S_{n+i\,i}$ implies that $L_{n+i\,i} = S_{n+i\,i}$. Thus $e_{ii} \in L$ if $i \in I \neq 2, 5$, because for those i, $S_{n+i\,i} S_{i\,n+i} = 0$. Hence S is a direct summand order. Now we consider elements $g(A, D)$ where $A = E + e_{ij}$, $i \neq j$, $i, j = 1, \ldots, n$, $a \in S_{ij}$. If $i \notin I = 2$, then $e_{ii} \in L$ and $e_{ii}\, g(A, D) = e_{ii} + a e_{ij}$, hence $a\,e_{ij} \in L$ and $L_{ij} = S_{ij}$. Similar argument holds for (i, j), $j \notin I = 2$. The σ-invariance of both L and S implies that they also coincide in the positions $(n + i, n + j)$ and $L_{ij}\, e_{ij} \subset L$ for those values of (i, j). Next from $g(A, D) - E = \theta_{ij} = e_{ij} - e_{n+j\,n+i} \in L$, $i \neq j$, $a = 1$, (i, j) in $(2, 2)$, we get $\theta_{ij}\, \theta_{ji} = e_{ii} + e_{n+j\,n+j} \in L$, i.e., $L_{ii} = S_{ii}$ for all $i = 1, \ldots, 2n$. Now by considering $g(H)$, $H = b e_{ij} + b e_{ji}$, $b \in S_{i\,n+j} = S_{j\,n+i}$, (respectively ${}^t g(H)$, $b \in S_{n+i\,j} = S_{n+j\,i}$), $(i, j) \in (2, 2)$, we see that $b\,e_{i\,n+j}$, $b e_{j\,n+i}$, (respectively $b\,e_{n+i,\,j}$, $b\,e_{n+j\,i}$) all lie in L, if either $i \notin I = 2$ or $j \notin I = 2$, or both. Hence L and S coincide in the positions $(i, n+j)$, and $(n+i, j)$, whenever $i \notin I = 2$ or $j \notin I = 2$, or both. Now if we observe that $e_{i\,n+j} = -e_{i\,n+i}\,\theta_{ji}$, $\pi e_{n+i\,j} = \pi\, e_{n+i\,i}\,\theta_{ij}$ both lie in L, for all $(i, j) \in (2, 2)$, we get that $L_{ij} = S_{ij}$ for all $(i, j) \in (I, J)$, $I, J = 2, 5$, and therefore $L_{ij} = S_{ij}$ for all $i, j = 1, \ldots, 2n$.

We wonld like to point out that this fact just proved does not mean that $L = S$, because we do not know whether or not $e_{ii} \varepsilon L$ for all $i \varepsilon I = 2, 5$. If we consider the order generated by L and anyone of these e_{ii}, we obtain a direct summand order, because $e_{ij} = e_{ii}\theta_{ij}$, $e_{ji} = \theta_{ji}e_{ii}$ lies in this order and hence the same happens for e_{jj} for all j in $I = 2$ and similarly for all $j \varepsilon I = 5$. Moreover this order has to be S, because two direct summand orders M and N coincide if and only if $M_{ij} = N_{ij}$ for all (i, j). The lack of knowledge that $L = S$ introduces some tecnical difficulties because we need that our orders be direct summand. We shall prove next that if M is a maximal order containing L, then M also contains S. If $L = S$ we are done, as well as in the case where $e_{ii} \varepsilon M$ for some $i \varepsilon I = 2, 5$ because $e_{n+j\, n+j} = \theta_{ij}\,\theta_{ji} \cdot e_{ii}$. Clearly $M_{ij} \supset L_{ij}$, for all (i, j). We shall assume that $e_{ii} \varepsilon M$ for all $i \varepsilon I = 2, 5$. If we omit from L the blocks (I, J) such that $I = 2, 5$, or $J = 2, 5$, or both, we obtain a direct summand order in $M_{4p}(k)$ which is L_p, hence it is maximal. Therefore M and S coincide for $(i,j) \varepsilon (I, J)$, $I, J = 1, 3, 4, 6$, i. e., $M_{ij} = L_{ij}$ for all such (i, j). Let $(i, j) \varepsilon (2, 1)$ (respectively $(3, 2)$, $(1, 5)$) and let $g \varepsilon M$, $g = (g_{ij})$ $g_{ij} \varepsilon M_{ij}$; ince $e_{jj}, e_{ti} \varepsilon L$, $(j, t) \varepsilon (2, 1), (5, 6)$) we get that $e_{ti}ge_{jj} = g_{ij\ tj}$ (respectively $e_{ii}ge_{jt} = g_{ij}e_{it}$), therefore $o = L_{ij} \subset M_{ij} \subset M_{tj} = o$ (respectively $o = L_{ij} \subset M_{ij} \subset M_{it} = o$) and consequently $M_{ij} = o = L_{ij}$ for all $(i, j) \varepsilon (2,1), (3, 2), (1,5)$. By the σ-invariance, $M_{ij} = L_{ij}$ for all $(i, j) \varepsilon (4, 5), (5, 6)$,

$(2,4)$. If there exists $g \in M$ such that $g_{ij} = 1$ for some (i,j) in $(1,2)$ (respectively in $(5,1)$), as e_{ji} and $e_{jj} + e_{tt}$, $(j,t) \in (2,5)$, both lie in L (respectively $e_{ii} + e_{ss} \in L$, $(i,s) \in (5,2)$), we get that $e_{ji}\, g(e_{jj} + e_{tt}) = e_{jj} + g_{it}\, e_{jt}$ (respectively $(e_{ii} + e_{ss})\, g\, e_{ji} = e_{ii} + g_{sj}\, e_{si}$). But $g_{it} \in M_{it} = 0$, $e_{jt} \in L$ (respectively, $g_{sj}\, e_{si} \in L$) consequently $e_{jj} \in M$ (respectively $e_{jj} \in M$), which is a contradiction. Therefore $M_{ij} = p = L_{ij}$ for all (i,j) lying either in $(5,1)$ or in $(1,2)$; the same argument applies to the positions (2.3) with the following slight modification : we consider $(e_{ii} + e_{tt})\, g\, e_{ji} = g_{ij}\, e_{ii} + g_{tj}\, e_{ti}$, $(i,j) \in (2,3)$, $(t,i) \in (5,2)$. If $p \ g_{tj}$, then $(g_{ij}\, e_{ii} + g_{tj}\, e_{ti})\ \ e_{it} = g_{tj}\, e_{tt} + g_{ij}\, e_{it}$, and as $e_{it} \in L$ we get that $g_{tj}\, e_{tt} \in M$ or $e_{tt} \in M$ which is a contradiction. As an immediate consequence of this argument we have that $g_{ij}\, e_{it} \in L$ if $(i,j) \in (2,3)$ hence $g_{tj} \in p$ or $M_{tj} = p$ for all $(t,j) \in (5,3)$. σ- invariance implies that same is true for $(5,4)$, $(4,2)$, $(6,2)$, and $(6,5)$. Next if $M_{ij} = p^{-1}$, $(i,j) \in (2,6)$, we let $g \in M$, such that $g_{ij} = 1/\pi$ and consider $(e_{ii} + e_{ss})\, g(\pi\, e_{ji})$, $(j,s) \in (6,5)$ to get $e_{ii} \in L$ which is a contradiction. The position $(3,5)$ will follow from the σ- invariance. Next if for some (i,j) in $(2,2)$ there exists g such that $g_{ij} = 1/\pi$, then we replace g by $\pi\, e_{ti} g\, e_{jt} = e_{tt}$, $(j,t) \in (2,5)$, which is a contradiction. Same argument applies to the entries in $(5,5)$. If this happens to $(i,j) \varepsilon (2,5)$, we use $\pi e_{ji} g(e_{ii}+e_{jj}) = = e_{jj} + \pi g_{ii} e_{ji}$ and from $g_{ii} \varepsilon O$ we get $e_{jj} \varepsilon M$. Finally if $g_{jj} = 1$ for some $(i,j) \varepsilon (5,2)$ we use $(e_{ii} + e_{jj}) g e_{ji} = e_{ii} + g_{jj} e_{ji}$ and apply the same argument as before. This concludes the proof that $M_{ij} = L_{ij}$ for all (i,j) and therefore $M \subset S$ which contradicts the fact that M is maximal.

If M is a maximal σ- invariant order containing L, then $M = End_{\sigma}(M)$, for some lattice M in V, hence $L \subset End_{\sigma}(M)$ or $gM \subset M$ for all $g \varepsilon L$. This suggests that our next step is the calculation of all lattices in V left invariant by L. As $e_{ii} \varepsilon M$ for all $i = 1, \ldots 2n$, we must have that if $x \varepsilon M$, $x = \Sigma x_i e_i$, then $x_i e_i \varepsilon M$. We set $M = M_1 e_1 \oplus \ldots \oplus M_n e_n$. By replacing M by aM if necessary we may assume all M_i integral. Next we observe that if $A e_{ij}, A^{-1} e_{ji} \subset M$, then $M_i = M_j$; hence $M_i = M_j$ for all $(i,j) \varepsilon (I,I)$, $I = 1, \ldots, 6$ and $(i,j) \varepsilon (1,4)$. We set $M_i = A_i$ if $i \varepsilon I = 1,2,3$ and $M_i = B_J$ if $i \varepsilon J = 4,5,6$. If we choose the scalar $\underline{a}$ such that $A_3 = 0$, then as $e_{ij} \varepsilon M$:, $(i,j) \varepsilon (3,2)$ and $(2,1)$, we get that $A_3 \supset A_2 \supset A_1$ and $\pi e_{ij} \varepsilon M$ $(i,j) \varepsilon (1,3)$ implies that $A_1 \supset pA_3$. We get from the σ- invariance and from $\pi e_{ij}, (1/\pi) e_{ji} \varepsilon M$, $(i,j) \varepsilon (6,3)$, that $A_1 = B_1 \supset B_2 \supset B_3 = pA_3 = p$. These conditions implies that we have only the following possibilities

1. $A_1 = A_2 = A_3 = B_1 = o$, $B_2 = B_3 = p$

2. $A_1 = A_2 = A_3 = B_1 = B_2 = o$, $B_3 = p$

3. $A_1 = B_1 = B_2 = B_3 = p$, $A_2 = A_3 = o$

3. $A_1 = A_2 = B_1 = B_2 = B_3 = p$, $A_3 = o$

These possibilities give respectively the following lattices: L_p, L_s, $g(p)L_s$, $g(p)L_p$. Hence only $\Delta(L_p)$ and $\Delta(L_s)$ contain Δ_s.

q. e. d.

We would like to point out that there exists an order L^* in $M_n(k)$, where $K = k(\sqrt{\pi})$ such that $L^* \cap M_n(k) = L$; if we denote by o^* the ring of integers of K, this order is $End_\sigma(L^*)$ where $L^* = M$ with $A_2 = A_3 = o^*$ and $A_1 = B_1 = B_2 = (\sqrt{\pi})$, $B_3 = p$. Clearly $L^* \cap V = g(p)\, L_s$ and $(1/\sqrt{\pi})\, L^* \cap V = L_s$.

Theorem 2. The number of conjugacy classes of maximal arithmetic groups in Sp_n is $n+1$, or equivalently, the number of conjugacy classes of maximal open compact subgroups of PSp_n is $n+1$.

Proof. It is well known that $N(\Delta(L_s))$, $0 \leqq s \leqq [\frac{n}{2}]$ are maximal, because $\Delta(L_s)$ is maximal in $Sp_n(k)$. They are now pairwise conjugate; for if $N(\Delta(L_s))$ is conjugate to $N(\Delta(L_t))$, then by lemma 2, $\Delta(L_s)$ is conjugate to $\Delta(L_t)$ in $Gp_n(k)$, hence there exists $h \in Gp_n(k)$ such that L_s leaves hL_t invariant. By similar argument to the proof of the above theorem we must have $h = E$, $t = s$. Next $N(\Delta_s$, $0 < s < [\frac{n}{2}]$, are maximal and not conjugate to one $N(\Delta(L'))$, otherwise $\Delta(L)$ and Δ_s would be conjugate; noreover they are not pairwise conjugate, for if $N(\Delta_s) = g^{-1}N(\Delta_t)g$ then $\Delta_s = g^{-1}\Delta_t g$ hence either $\Delta(L_t)$ and $\Delta(L_s)$ are conjugate, or $\Delta(L_t)$ and $\Delta(L_{n-s})$ are conjugate, or $t = s$. If $n = 2m$ is even, then we have $m+1$ groups in the first group and m groups in the second group; altogether $2m+1 = n+1$. If $n = 2m+1$ is odd we have $m+1$ groups in the first group and $m+1$ in the second; altogether $n+1$.

q. e. d.

Closing this paper we remark that Shimura [7] proved that $Gp_n(k) = \sqrt{}\Gamma(L_o)\, D\, \Gamma(L_o)$ where D is the set of diagonal matrices in $Gp_n(k)$, i.e., $\{Gp_n, \Gamma(L_o)\}$ has the elementary divisor property. This is no longer true for L_s, $s \neq 0$, because if $g(p)$ can be diagonalized , say $g(p) = ghg'$, g , $g' \in \Gamma(L_S)$, then

$N(\Delta(L_S) \cap \Delta(hL_S))$ is maximal by theorem 1 ; on the otherhan we may

assume h = diagonal $\{E, \pi E\}$ and $hL_S = g(0)L_S$ and

$N(\Delta(0))$ is not maximal by corolary of lemma 8 ; this is a contradiction. Therefore $g(p)$ cannot be diagonalized by the operations of $\Gamma(L_S)$.

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