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ON KAN'S CONDITION

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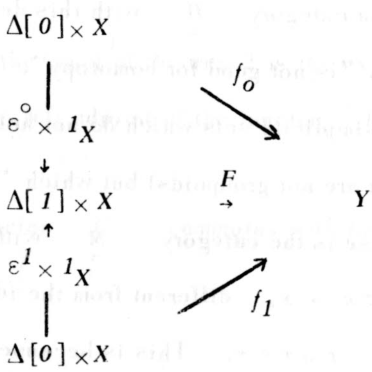
Introduction : In the category $\underline{\underline{\Delta^\circ Ens}}$, of the simplicial sets, there exists a "Unit interval" : the simplicial set $\Delta[1]$, with extremities ε^i : $\varepsilon^i : \Delta[0] \to \Delta[1]$ $(i = 0, 1)$ ([1], [2]).

On the other hand, since in $\underline{\underline{\Delta^\circ Ens}}$ the products are representable and $\Delta[0]$ is a final object, these exists in $\underline{\underline{\Delta^\circ Ens}}$ a completely natural notion of homotopy : two arrows $f_o, f_1 : X \to Y$ are "simply homotopic" if there is an arrow F which makes the following diagram commutative.



However, and contrarily to what happens in the Topologycal case, the relation of "Simple homotopy", is not, in general, an equivalence relation on the set $Hom(X, Y)$, unless Y satisfies extra conditions (for example, Kan's extension condition (Cf. [1])).

Therefore, Kan's condition is a sufficient condition on Y, in order for the relation of simple homotopy to be an equivalence relation on $Hom(X, Y)$, for every X.

The purpose of this paper is to show that Kan's condition is not necessary.

In order to exhibit a counter-example in which the simple homotopy is not an equivalence relation and Kan's condition is not necessary, recall that all categories are in some way included in the category of simplicial sets (§ 1). This identification establishes a one-to-one correspondence between natural transformations and simple homotopies. Furthermore, in order for a category $\underset{=}{C}$ to be a groupoid a necessary and sufficient condition is that the simplicial set $D(\underset{=}{C})$ to which $\underset{=}{C}$ is identified fulfills Kan's condition ([3] and T. 2).

Since it is evident that the relation : *(h)* « there exists a natural transformation between the functors F and G from $\underset{=}{A}$ to $\underset{=}{B}$ », is not in general symetric, an example of a category $\underset{=}{B}$ with this defect provides a simplicial set $D(\underset{=}{B})$ which "is not good for homotopy". That is why we can restrict ourselves to look for simplicial sets which do not hold Kan's condition (for example categories which are not groupoids) but which "are good for homotopy". The example that we use is the category $\underset{=}{M}$ with only one object x, and only one arrow $r : x \to x$, different from the identity on x, which satisfies the relation $r \circ r = r$. This is because :

1) certainly $\underline{\underline{M}}$ is not a groupoid

2) $\underline{\underline{M}}$ satisfies the following property : if in a diagram with f and g arbitrary

the vertical arrows are both equal to r, then the diagram commuts.

Therefore, given $F, G : \underline{\underline{A}} \to M$ any functors, there always exists a natural transformation between then $\lambda : F \to G$, $\lambda_Y = r$; $F(Y) = x$, $G(Y) = x$.

This implies that on Funct $(\underline{\underline{A}}, \underline{\underline{M}})$ the relation (h) is an equivalence relation .

Theorem 2 alauds us to conclude that $D(\underline{\underline{M}})$ is good for homotopy.

Theorem 2. *Let* $\underline{\underline{A}}$ *be a category. If for each category* $\underline{\underline{C}}$ *the relation* (h) *on the set Funct* $(\underline{\underline{C}}, \underline{\underline{A}})$ *is an equivalence relation, then for every simplicial set* X , *the relation of simple homotopy on the set* $Hom(X, D(A))$ *is also an equivalence relation.*

An essencial fact on the proof of theorem 2 is the existence of a functor $G : \underline{\underline{\Delta^\circ Ens}} \to \underline{\underline{Cat}}$, left adjoint of the functor D, for which we will show :

Theorem 1. *The functor* G *commutes with finite products.*

§ 1. THE FUNCTOR D

Let us recall some of the properties of the functor

$$D : \underline{\underline{Cat}} \to \underline{\underline{\Delta^\circ Ens}}$$

wich can be found in [1], [3], [4].

First D is a fully faithful functor, which means that the application

$$D : Funct(\underline{\underline{A}}, \underline{\underline{B}}) \to Hom(D(\underline{\underline{A}}), D(\underline{\underline{B}}))$$

induced by D, is an isomorphism.

Let J_n be the category whose objects are the integer $0, 1, \ldots, n$, and in which there is one ,and only one arrow from i into j if $i \leqq j$

The second characteristic of D is that it establishes a one-to-one correspondence between the categories J_n and the simplicial sets of the type $\Delta[n]$ $(n \geqq 0)$:

$$D(J_n) = \Delta[n].$$

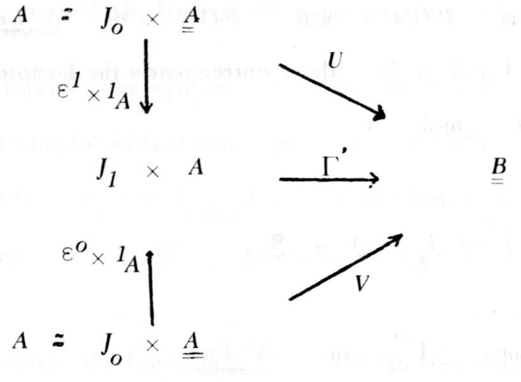
Even more : to an increasing function $W' : [n] \to [m]$ $([n] = \{0, 1, \ldots, n\})$, there is associated, in an obvious manner, a functor $W' : J_n \to J_m$ and a simplicial function $W'' : \Delta[n] \to \Delta[m]$, for which $D(W') = W''$ holds.

Another property which will be usefull is the correspondence which D establishes between natural transformation and simple homotopies.

Before we enunciate this correspondence, recall that for any natural transformation $\Gamma : U \Rightarrow V$, between two functors $U, V : \underline{A} \to \underline{B}$, there corresponds a functor, called homotopy of $\underline{\underline{Cat}}$

$$\Gamma' : \underline{J}_1 \times \underline{A} \to \underline{B}$$

and a commutative diagram



Γ' is defined as follows :

$$\Gamma'(0, X) = U(X)$$

$$\Gamma'(1, X) = V(X), \quad X \text{ an object of } \underline{A} .$$

For an arrow of the kind $(id_0, f) : (0, X) \to (0, Y)$. $\Gamma'(id_0, f) = U(f)$.

Similarly $\Gamma'(id_1, f) = V(f)$. For an arrow of the kind $(\delta, f) : (0, X) \to (1, Y)$,

$$\Gamma'(\delta, f) = V(f) \circ \Gamma_X : U(X) \to V(Y) = \Gamma_Y \circ U(f) : U(X) \to U(Y) \to V(Y) .$$

Inversely for any homotopy $\Gamma' : U \sim V$ between functors there corresponds a natural transformation $\Gamma : U \Rightarrow V, \Gamma_X : U(X) \to V(X)$ given by $\Gamma'(\delta, 1_X) = \Gamma_X$, establishing an isomorphism between the set $Trans(U, V)$ of natural transformations from U into V, and the set $Homot(U, V)$ of homotopies from U into V, in $\underline{\underline{Cat}}$.

Since D commutes with the products (because of the existence of a left ad - joint functor) an is fully faithfull, it establishes a one-to-one correspondence between the natural transformation from U into V $(U, V : \underline{\underline{A}} \to \underline{\underline{B}})$, and the simple homotopies from $D(U)$ into $D(V)$ in $\underline{\underline{\Delta^\circ Ens}}$: to a natural transformation $\Gamma : U \to V$ there corresponds the homotopy Γ' in Cat between U and V

$$\Gamma' : J_1 \times \underline{\underline{A}} \to \underline{\underline{B}}$$

which induces the homotopy Γ'' in $\underline{\underline{\Delta^\circ Ens}}$

$$D(J_1 \times A) \xrightarrow{D(\Gamma')} D(B)$$

$$\wr\wr$$

$$D(J_1) \times D(A)$$

$$\wr\wr$$

$$\Delta[1] \times D(\underline{\underline{A}})$$

§ 2. THE FUNCTOR G

2.1. The purpose of this paragraph is to show that the functor

$$G : \underline{\underline{\Delta^\circ Ens}} \to \underline{\underline{Cat}}$$

the left adjoint functor of D (which is constructed for example in [1]) commutes with finite products.

2.2. Let us establish, first of all, some notations which will be usefull in § 3.

The adjoint isomorphism $\varphi : Func\,(G(X), A) \to Hom\,(X, D(\underline{\underline{A}}))$ provides a simplicial function $\psi_X : X \to D\,G(X)$ (Taking $\underline{\underline{A}} = G(X)$), given by $\psi_X = \varphi(1_{G(X)})$, and a function $\theta_A : (GD, (\underline{\underline{A}})) \to \underline{\underline{A}}$ (taking $X = D(\underline{\underline{A}})$) given by $\theta_A = \varphi^{-1}(1_{D(\underline{\underline{A}})})$.

Conversely the function φ (Resp. its inverse) is obtained from ψ (Resp. from θ) as follows : if $\alpha \in Func(G(X), \underline{\underline{A}})$ (Resp. $\beta \in Hom(X, D(A)))$, $(\alpha) = D(\alpha) \circ \psi_X$ (Resp. $\varphi^{-1}(\beta) = \theta_A \circ G(\beta)$).

2.3.1. The functor G is characterized by the following two properties :

1) If $W : \Delta[n] \to \Delta[m]$ and if $W' : J_n \to J_m$ is the associated functor (§ 1), then $G(W) = W'$.

2) G commuts with right hand limits.

This is so because if these two conditions holds then we proceed as follows [1].

2.3.2. Let X be a simplicial set. Let $\Delta/_X$ be the category whose objects are the arrows $f : \Delta[n] \to X$, and in which the morphisms from $f : \Delta[n] \to X$ into $g : \Delta[m] \to X$ are the simplicial functions $W : \Delta[n] \to \Delta[m]$ such that $g \circ W = f$.

We denote $s_X : \Delta/_X \to \underline{\underline{\Delta^\circ Ens}}$ for the functor "source" which associated with $f : \Delta[n] \to X$ the simplicial set $\Delta[n]$ and with $W : f \to g$ it associates the simplicial function $W : \Delta[n] \to \Delta[m]$.

Now we can recall :

Lemma 1. For each simplicial set X, $\varinjlim s_X$ is representable. That is to say, it is defined in $\underline{\underline{\Delta^\circ Ens}}$ and furthermore, $\varinjlim s_X \simeq X$. If G commutes with right hand limits, then

$$G(X) = \varinjlim (G \circ s_X).$$

Lemma 2. For each simplicial set Y, let

$$c_Y : \Delta/_Y \to \underline{\underline{\Delta^\circ Ens}}$$

denote the constant functor of value Y. Then ([1])
$Y \times X \simeq \varinjlim (c_Y \times s_X)$.

2.3.3. Let $F_X : \Delta/_X \to \underline{\underline{Cat}}$ be the functor which associates to each $f : \Delta[n] \to X$ the category J_n and to each

$W : f \to g$ the functor $W' : J_n \to J_m$. Then we have :

$$G(X) \simeq G(\lim_{\to} s_X) \quad \text{(Lemma 1)}$$

$$\simeq \lim_{\to} (G \circ s_X) \quad (G \text{ commutes with right hand limits})$$

$$\simeq \lim_{\to} F_X \quad \text{(Property 1, 2.3.1)}$$

2.3.4. We can say more about G : The adjointing morphism $\theta_A : G(D(A)) \to \underline{A}$ is an isomorphism.

2.3.5. Now we can prove theorem 1.

First part : $G(\Delta[n] \times \Delta[m]) \simeq G(\Delta[n] \times G(\Delta[m])$

In fact : $G(\Delta[n] \times \Delta[m]) \simeq G(D(J_n) \times D(J_m))$ (§ 1)

$$\simeq G(D(J_n \times J_m)) \quad (\S\ 1)$$

$$\simeq J_n \times J_m \quad (2.3.4.)$$

Second part : $G(\Delta[n] \times X) \simeq G(\Delta[n]) \times G(X)$.

In fact $G(\Delta[n] \times X) \simeq G(\lim_{\to} c_{\Delta[n]} \times s_X)$

$$\simeq \lim_{\to} (G \circ (c_{\Delta[n]} \times s_X))$$

But analysing the functor $G \circ (c_{\Delta[n]} \times s_X)$ it can be seen, with the helps of the first part, that it is isomorphic to the functor $c_{J_n} \times G \circ s_X$. Then

$$
\begin{aligned}
G(\Delta[n] \times X) &\simeq \varinjlim (G \circ (c_{\Delta[n]} \times s_X)) \\
&\simeq \varinjlim (c_{J_n} \times G \circ s_X) \\
&\simeq J_n \times \varinjlim (G \circ s_X) \\
&\simeq J_n \times G(\varinjlim s_X) \\
&\simeq J_n \times G(X) \\
&\simeq G(\Delta[n]) \times G(X) .
\end{aligned}
$$

Third part : $G(X \times Y) \simeq G(X) \times G(Y)$

In fact :

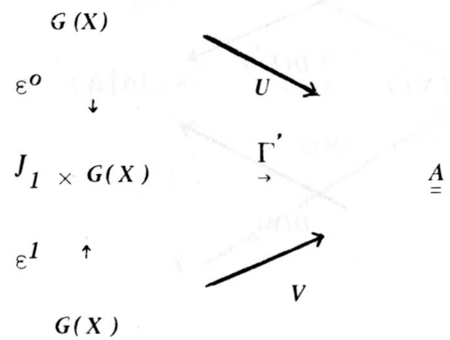
$$
\begin{aligned}
G(X \times Y) &\simeq G(X \varinjlim s_Y) \\
&\simeq G(\varinjlim (c_X \times s_Y)) \quad \text{(lemma 2)} \\
&\simeq \varinjlim G(c_X \times s_Y) \\
&\simeq \varinjlim (G \circ c_X \times G \circ s_Y) \quad \text{(Consecuence of second part)} \\
&\simeq \varinjlim (c_{G(X)} \times G \circ s_Y) \\
&\simeq G(X) \times \varinjlim G \circ s_Y \\
&\simeq G(X) \times G(\varinjlim s_Y) \\
&\simeq G(X) \times G(Y)
\end{aligned}
$$

§ 3. PROOF OF THEOREM 2

3.1. *Lemma.* Let X be a simplicial set and $\underline{\underline{A}}$ be a category. Then the adjointing isomorphism $\varphi : Funct\,(G\,(X),\ \underline{\underline{A}}) \tilde{\to} Hom\,(X\,, D(\underline{\underline{A}}))$

assingns the homotopy relation in $Hom\,(X\,, D\,(\underline{\underline{A}}))$ to the relation *(b)* in $Funct\,(G\,(X),\ \underline{\underline{A}})$, that is to say there exist a natural transformation $\Gamma : U \Rightarrow V$ between two functor $U\ V : G\,(X) \to \underline{\underline{A}}$ if and only there exists a simple homotopy from $\varphi(U)$ to $\varphi(V)$.

Proof. Let $U, V : G\,(X) \to \underline{\underline{A}}$ be functors such that there exists a natural transformation $\Gamma : U \Rightarrow V$. According with § *1* there exists a homotopy $\Gamma' : J_1 \times G\,(X) \to \underline{\underline{A}}$ such that the following diagram commutes :



Applying D to this diagram we get the following commutative one :

$$\begin{array}{ccc}
D(G(X)) & & \\
D(\varepsilon^{1})\downarrow & \searrow^{D(U)} & \\
D(J_1 \times G(X)) & \xrightarrow{D(\Gamma')} & D(\underline{\underline{A}}) \\
D(\varepsilon^{1})\uparrow & \nearrow_{D(V)} & \\
D(G(X)) & &
\end{array}$$

on equivalently (§ 1)

$$\begin{array}{ccc}
D(G(X)) & & \\
\varepsilon^{o}\downarrow & \searrow^{D(U)} & \\
\Delta[1] \times D(G(X)) & \xrightarrow{D(\Gamma')} & D(\underline{\underline{A}}) \\
\varepsilon^{1}\uparrow & \nearrow_{D(V)} & \\
D(G(X)) & &
\end{array}$$

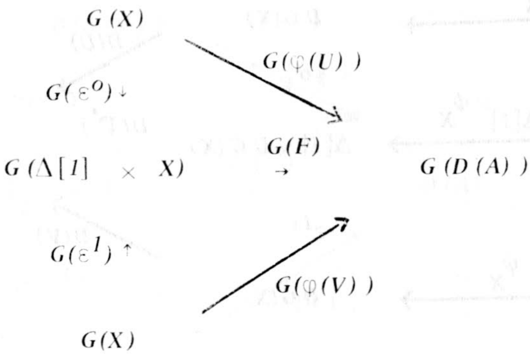
Let $\psi_X = \varphi(1_{G(X)}) : X \to DG(X)$ be the adjointing isomorphism (§ 2, 2.2). Since $\varphi(U) = D(U \circ \psi_X$ and $\varphi(V) = D(V) \circ \psi_X$, then the diagram.

$$
\begin{array}{ccccc}
X & \xrightarrow{\psi_X} & D\,G(X) & & \\
\downarrow & & \varepsilon^{o}\downarrow & \searrow^{D(U)} & \\
\Delta[1]\times X & \xrightarrow{1_{\Delta[1]}\times\psi_X} & \Delta[1]\times D\,G(X) & \xrightarrow{D(\Gamma')} & D(\underline{\underline{A}}) \\
\uparrow & & \varepsilon^{1}\uparrow & \nearrow_{D(V)} & \\
X & \xrightarrow{\psi_X} & D\,G(X) & &
\end{array}
$$

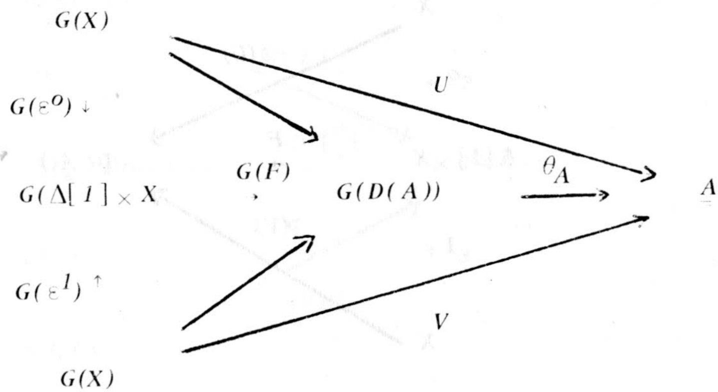
establishes a homotopy from $\varphi(U)$ to $\varphi(V)$. Conversely : Let F be a homotopy from $\varphi(U)$ to $\varphi(V)$

$$
\begin{array}{ccc}
X & & \\
\varepsilon^{o}\downarrow & \searrow^{\varphi(U)} & \\
\Delta[1]\times X & \xrightarrow{F} & D(\underline{\underline{A}}) \\
\varepsilon^{1}\uparrow & \nearrow_{\varphi(V)} & \\
X & &
\end{array}
$$

Applying G we obtain :



Let θ_A be the adjointing isomorphism (§ 2, 2.2). The diagram above induces, by composition, the following one :



Because of the commutativity of G , proved in § 2 , Thm. 1, this diagram takes the form :

$$\begin{array}{ccc} G(X) & & \\ & \searrow^{U} & \\ \varepsilon^{0} \downarrow & & \\ J_1 \times G(X) & \xrightarrow{\omega} & A \\ \varepsilon^{1} \uparrow & & \\ & \nearrow_{V} & \\ G(X) & & \end{array}$$

Where $\omega = \theta_A \circ G(F)$ composed with the canonic isomorphism $G(\Delta[1] \times X) \cong J_1 \times G(X)$.

To this homotopy corresponds according with § *1*, a natural transformation Γ from U into V.

Note : We have realy proved more than asked in our lemma. We have proved that the function $f : Trans(U, V) \to Hom(\varphi(U), \varphi(V))$ which assigns to each natural transformation Γ from U to V the homotopy $D(\Gamma') \circ (1_{\Delta[1]} \times \psi_X)$, is an isomorphism.

3.2. Proof of Theorem 2.

Assume that for each category $\underline{\underline{C}}$ the relation (*b*) in $Funct(\underline{\underline{C}}, \underline{\underline{A}})$ is an equivalence relation. Let X be any simplicial set. From the isomorphism $Hom(X, D(\underline{\underline{A}})) \cong Funct(G(X), \underline{\underline{A}})$ and the lemma before, it can be concluded that, since in $Funct(G(X), A)$, (*b*) is an equivalence relation, so is the simple homotopy relation in $Hom(X, D(\underline{\underline{A}}))$.

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