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SECONDARY INVARIANTS FOR LINKS

by

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Dedicated to Professor H. Yerly

1. Let mS^1 be the space consisting of m disjoint (oriented) copies of the circle. An oriented m-link is a (polygonal) embedding $l : mS^1 \to S^3$. Let $X = S^3 - Im(l)$ and π be its fundamental group.

Associated with a projection of the link l, there is a presentation (cf. [3 ; ch. I]) of the group π called the Wirtinger presentation :

$$\pi = < x_{ij} \mid r_{ij}, \ 1 \leq i \leq m \ ; \ 1 \leq j \leq \lambda_i >$$

where x_{ij} is represented by a loop going once around an arc of the i^{th} component l_i in the positive direction and $r_{ij} = u_{ij} x_{ij} u_{ij}^{-1} x_{i,j+1}^{-1}$, $u_{ij} = x_{pq}^{\varepsilon}$, $\varepsilon = \pm 1$.

Now consider a set of elements s_{ij} defined by $s_{ij} = v_{ij} x_{il} v_{ij}^{-1} x_{i,j+1}^{-1}$ where $v_{ij} = u_{ij} u_{i,j-1} \dots u_{il}$. Then the following is a presentation of π :

$$\pi = < x_{ij} \mid s_{ij} >$$

Finally write $x'_{ij} = x_{ij} x_{i1}^{-1}$ and $x_i = x_{i1}$ $(i = 1, \ldots, m;\ j = 1, \ldots, \lambda_i)$. The group π can be presented by

(1) $$\pi = < x_1, \ldots, x_m, x'_{ij} \mid x'_{i,j+1} = [v_{ij}, x_i], [v_{i\lambda_i}, x_i] = 1 >$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, \lambda_i - 1$. The statements above are proved in [4 ; § 2].

Let $F(m)$ be the free group in the letters $x_1, \ldots, x_m$ and $i : F(m) \to \pi$ the map given by $x_\mu \mapsto x_\mu$.

Observe that to obtain presentation (1), we have made the following choices:

a) a projection of the link l ;

b) a choice of x_{i1} for $i = 1, \ldots, m$.

Once these choices have been made, the group

(2) $$\pi^* = < x_1, \ldots, x_m, x'_{ij} \mid x'_{ij+1} = [v_{ij}, x_i] >$$

is determined. There is a canonical epimorphism $\beta : \pi^* \to \pi$.

2. In [5] some numerical invariants $\bar{\mu}$ are defined.

LEMMA (1). If all the numbers $\bar{\mu}$ are zero, the map i induces isomorphisms $i : F/F_n \cong \pi/\pi_n$ for all finite n.

Here F stands for $F(m)$ and given a group G, G_n is the n^{th} member of the lower central series of G. The purpose of this note is to define and describe certain new (secondary) invariants for links that can be constructed when the (primary) invariants $\overline{\mu}$ vanish. These new invariants will turn out to be the obstructions for l to be a boundary link (cf. [7]).

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COROLLARY (2). The map i *induces an embedding* $F \subset \pi / \pi_\omega$, *where* $\pi_\omega = \bigcap \pi_n$.

LEMMA (3). The group π^* *verifies*

i) $H_1(\pi^*) = \mathbf{Z}^m$;

ii) $H_2(\pi^*) = 0$, *in fact* π^* *has a presentation (namely (2)) with defect* m;

iii) π^* *has weight* m, *that is*, π^* *is normally generated by* m *elements*.

Proof : Statement i) is obvious and iii) follows from the fact that if we add the relations $x_i = 1$ to presentation (2), the remaining relations yield $x'_{ij} = 1$. Assertion ii) follows from [2 ; p. 106] .

COROLLARY (4). The map i *induces*

$$i : F/F_n \cong \pi^*/\pi^*_n$$

$$i : F \subset \pi^*/\pi^*_\omega .$$

The proof follows from Theorem 3.4 of [8] and Lemma (3). Further :

COROLLARY (5). The map β *induces isomorphisms*

$$\pi^*/\pi^*_n \approx \pi/\pi_n \quad \text{for} \quad 2 \leq n \leq \omega$$

This follows from Lemma (1), Corollary (4) and Theorem 3.4 of [8].

Observe that the sequence F/F_n forms, with the obvious mappings, an inverse system. Thus we can form the group $\overline{F} = Inv.lim.(F/F_n)$ a totally disconnected compact group. Since $F_\omega = 1$, F is embedded in $\overline{F}$. In [1] we find the following description of $\overline{F}$: let P be the ring of formal integral series in the non-commuting variables $\alpha_i = x_i - 1$ $(i = 1, \ldots, m)$. Define the norm of a non-trivial series to be $1/n$ where n is the smallest integer for which there is a term of degree n with non-vanishing coefficient; for the trivial series define the norm to be zero. P is then a metric ring; let $\mu : \mathbf{Z}[F] \to P$ be given by

$$w \to w^o + \Sigma(\partial^n w/\partial x_{c_1} \ldots \partial x_{c_n})^o \alpha_{c_1} \ldots \alpha_{c_n}$$

where the coefficients are the partial derivatives of free differential calculus. The map μ is an embedding and the closure (in the given topology) of $\mu(F)$ is isomorphic to $\overline{F}$. An integral series $1 + \Sigma\lambda_{c_1 \ldots c_n} \alpha_{c_1} \ldots \alpha_{c_n}$ in P belongs to the closure of $\mu(F)$ if and only if, for all pairs (a,b) of sequences of the numbers $1, \ldots, m$ (perhaps repeated)

$$\lambda_a \lambda_b = \Sigma\, \mu(c) \lambda_c$$

where c ranges over all the results of infiltrating a and b and $\mu(c)$ is the Moebius number. Corollaries (4) and (5) can be rewritten as

COROLLARY (6). There exist finitely many elements b_j *of* $\overline{F}$, *such that* π/π_ω *is the subgroup of* $\overline{F}$ *generated by* F *and the* b_j.

If we write presentation (2) as the preabelian presentation (cf. [3; p.142])

$$(2') \quad \pi^* = \langle x_1, \ldots, x_m, b_1, \ldots, b_r \mid b_j = B_j(x_\mu, b_\nu), \mu = 1, \ldots, m; j, \nu = 1, \ldots, r \rangle$$

where b_j **is one of the** x'_{ij} **and** B_j **has the form** $[v_{ij}, x_i]$, **then the elements** $b_j \bmod \pi_\omega$ **can be described in terms of its integral series in the closure of** $\mu(F)$ **as described above :**

LEMMA (7). Given π^* *presented by (2'), the element* b_j *mod* π^*_ω *has a series expansion where the first coefficients are given by*

$$(\partial b_j / \partial x_i)^o = 0$$

$$(\partial^2 b_j / \partial x_{i_1} \partial x_{i_2})^o = (\partial^2 B_j / \partial x_{i_1} \partial x_{i_2})^o$$

$$(\partial^3 b_j / \partial x_{i_1} \partial x_{i_2} \partial x_{i_3})^o = (\partial^3 B_j / \partial x_{i_1} \partial x_{i_2} \partial x_{i_3})^o +$$

$$+ \Sigma_k [(\partial^2 B_j / \partial b_k \partial x_{i_3})^o (\partial^2 B_k / \partial x_{i_1} \partial x_{i_2})^o]$$

where we use the notation $b_j \bmod \pi^*_\omega = 1 + \Sigma (\partial^n b_j / \partial x_{i_1} \ldots \partial x_{i_n})^o \alpha_{i_1} \ldots \alpha_{i_n}$.

The proof of this lemma is a straightforward computation,

3. The coefficients above defined depend of course on choices a) and b)

as defined in 1; a different choice will reflect certain numerical changes in the coefficients. By [6], the presentation (2') can be changed by a different choice of projection in the following way:

a1) add a generator b and a relation $b = b_j$ (some j);

a2) add two generators b, b' and relations $b' = b_j$ and $b = (b_j)^{b_k x_a}$;

a3) change a generator from b_j to b_j^w for some w.

Operations a1) to a3) correspond to operations $\Omega 1$ to $\Omega 3$ of [6; p.7]. A change of the choice of x_{i1} simply conjugates the generators.

In practice, for computational purposes, we can approximate $b_j \bmod \pi_\omega$ by $b_j \bmod \pi_n$ for a large n. This allows us to define $\nu(i_1, \ldots, i_s, r)$ as the $(i_1, \ldots, i_s)$ coefficient of $b_r \in \overline{F}(r, s = 1, 2, 3 \ldots)$. In order to extract invariants from the above coefficients we have to consider the sets $\{\nu(i_1, \ldots, i_s)\}$ modulo the indeterminacy introduced by operations a1), a2) and a3).

PROPOSITION (8). The sets

$S(i,j) = \{\nu(i,j,k),\ k \in N\}$ $(\nu(i,j,k) = 0$ *for large* $k)$

$S(i,j,k) = \{\nu(i,j,k,p) \bmod (\nu(i,j,p) - \nu(j,k,p)) \mid p \in N\}$

$S(i,j,k,p) = \{\nu(i,j,k,p,q) \bmod \Sigma_n [\nu(i,j,n)\nu(k,p,q) - \nu(i,j,q)\nu(k,p,n) +$

$+ \nu(i,p,q)\,\nu(j,k,n)] + \nu(j,k,p,q) - \nu(i,j,k,q)\}$

are invariants of the link l defined when the Milnor invariants $\overline{\mu}$ are all zero.

4. In preparing this section the author was helped by Mr. R. Peña of Yeshiva University, who kindly wrote a computer program for Example 1.

Example 1. In [7] it is proven that *homology* boundary links have zero Milnor invariants; for the example found in [7; p. 71] the first secondary invariants are :

$S(1,1) = \{1, -1, -4, -7, -8\}$

$S(2,2) = \{1, 4, 5, 8, 9, 12, 13, 16, 17\}$

$S(1,2) = \{1, -1, -3, -6, -10, -13, -15, -16\}$

$S(2,1) = \{1, 3, 6, 9, 12\}$.

Example 2. Let l be a boundary link. In that case there is a map $\pi/\pi_\omega \to F$ sending meridians to generators and the remaining generators in presentation (2') to elements of the form 1 or $[x_j^\varepsilon, x_i]$ $(\varepsilon = \pm 1)$. This assertion follows from considering a special projection of the Seifert surfaces viewed as disks with twisted bands attached. The secondary invariants will then be :

$S(i,j) \subset \{0, 1, -1\}$

$S(i, j, k) = \{0, 1 \mod 2\}$

$S(i_1, \ldots, i_4) = \{1 \mod 2\}$ for one arrangement $i_1, \ldots, i_4$ and $\{0\}$ for all other. $S(i_1, \ldots, i_r) = \{0 \mod 1\}$ for $r \geq 5$. This assertion follows from studying the series expansion of the elements of the form $[x_j^\varepsilon, x_i]$.

For links of multiplicity 2 (and probably for all multiplicities), it is useful to study the composite of the map $\mu : \mathbf{Z}[F] \to P$ and the map $P \to P/C$ where

C is the ideal generated by the α_i^2. In fact, C is a closed ideal and the composite is a monomorphism (cf. [3; p. 315]). In that case the inverse of $1+\alpha_i$ is $1-\alpha_i$ and the image of F is made out of polynomials.

For multiplicity 2 if l satisfies

$S(1,2) = S(2,1) = \{0, \pm 1\}$, $S(2,1,2) = S(1,2,1) \subset \{0,1 \bmod 2\}$ $S(i_1, \ldots, i_4) \subset \{0,1 \bmod 2\}$ and all other $S(i_1, \ldots, i_r) = \{0\}$, then it is a straightforward computation to verify that in P/C and for arbitrary p, it is possible to arrange the projection of the link to have $\nu(1,2,p) = 1$, $\nu(2,1,p) = -1$. Therefore,

(3) $-1 = \nu(1,2,p)\nu(2,1,p) = \nu(1,2,p) + \nu(2,1,2,p) + \nu(2,1,2,1,p) + \nu(2,1,1,2,p) + \nu(1,2,2,1,p) + \nu(1,2,1,2,p)$.

If $\nu(1,2,1,p) = -1$ and $\nu(2,1,2,1,p) \neq 0$ then, in order to verify (3), it is necessary that $\nu(2,1,2,p) = -1$, $\nu(1,2,1,2,p) = 0$, since in P/C, $\nu(2,1,1,2,p)$ and $\nu(1,2,2,1,p)$ are zero.

Since the higher invariants are zero, we conclude $b_p = [x_2^{-1}, x_1]$. Similarly for all other possible combinations.

THEOREM (9). A 2-link is boundary if and only if its Milnor invariants vanish and its secondary invariants are given by the formulas of Example 2.

5. Naturally, it is valid to conjecture that Theorem (9) is valid for m-links in general. The proof probably involves a more delicate analysis of the Shuffle relations of [1].

It is known [4], that the $\bar{\mu}$ are invariants of cobordism; the secondary invariants are not. We wonder if it is possible by cobordism to reduce the secondary invariants of a link with zero Milnor invariants to one with the invariants given in Example 2. This would lead the following:

CONJECTURE. If a link has zero Milnor invariants then it is cobordant to a boundary link.

In higher dimensions (cf. [2]), links *always* have zero Milnor invariants. Further every higher dimensional boundary link is split-cobordant (this is probably false in the classical case). Is it possible to cobord a given link to a boundary link? This leads to our second

CONJECTURE. Every higher dimensional link splits up to cobordism.

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