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On Spectral Compactness of Von Neumann Regular Rings

Sobre la compacidad espectral de los anillos regulares de von Neumann

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Abstract. We characterize the spectral compactness of commutative von Neumann regular rings. We show that through a process of adjunction of identity, we can obtain the Alexandroff compactification or a star compactification of the prime spectrum of certain von Neumann regular rings.

Key words and phrases. Spectral compactness, Prime spectrum, Boolean ring, Von Neumann regular ring, Compactification.

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Resumen. Caracterizamos la compacidad espectral de los anillos regulares de von Neumann conmutativos. Mostramos que a través de un proceso de adjunción de unidad, podemos obtener la compactación de Alexandroff o una compactación estelar del espectro primo de ciertos anillos regulares de von Neumann.

Palabras y frases clave. Compacidad espectral, espectro primo, anillo de Boole, anillo regular de von Neumann, compactación.

1. Introduction

Throughout this paper the rings considered are commutative, but not necessarily with identity and compact spaces are not necessarily Hausdorff. Von Neumann regular rings were introduced in 1936 by John von Neumann in [20], as an algebraic tool for studying certain lattices. These rings have been widely studied to the point that well-developed theories exist about them (see for example [3, 4, 8, 10, 14, 18]). Von Neumann regular rings are also known as absolutely flat rings, because of their characterization in terms of modules. Although a regular ring in von Neumann sense is not necesarily commutative, in this paper we will restrict ourselves to commutative rings. We are using the expression *von Neumann regular ring* (or simply *von Neumann ring*) to refer to a commutative ring such that, given an element a, there exists an element b such that $a = a^2 b$. This is equivalent to every principal ideal being generated by an idempotent.

Here are some examples of von Neumann rings:

- Every field is a von Neumann ring.
- We say that a ring is an *exp-ring* if for each element a there exists $n(a) \in \mathbb{Z}^+ \setminus \{1\}$ such that $a^{n(a)} = a$. Every exp-ring is a von Neumann ring. In fact, for each a in the ring we can take $b = a^{n(a)-2}$, if $n(a) > 2$ or $b = a$, if $n(a) = 2$. In [9] and in [13] we can find proofs about the commutativity of exp-rings.
    i) A ring A is a $p-$*ring* if it satisfies $pa = 0$ and $a^p = a$, for each $a \in A$, where p is a prime number. In particular for $p = 2$ we obtain the Boolean rings. Every $p-$ring is an exp-ring.
    ii) The ring $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ is a $p-$ring, for each prime p.
    iii) The ring $\mathbb{Z}_6$ is an exp-ring of characteristic 6 and $n(a) = 3$, for each a.
    iv) We denote $\mathcal{P}$ the set of prime numbers and consider the sub-ring of $\prod_{p\in\mathcal{P}} \mathbb{Z}_p$ whose elements are sequences with finitely many non-zero terms. This is an exp-ring such that $n(a) = 1 + \prod_{k=1}^{m}(p_k - 1)$, where $a_{i_1}, a_{i_2}, \dots, a_{i_m}$ are the non-zero terms of the sequence a, belonging to $\mathbb{Z}_{p_1}, \dots, \mathbb{Z}_{p_m}$ respectively.
- If A is a von Neumann ring and X is a set then A^X is a von Neumann ring.
- Any product of von Neumann rings is a von Neumann ring.
- Every quotient of a von Neumann ring is a von Neumann ring.

The last three items can be checked directly from the definition of von Neumann ring.

In the second Section we mention some known concepts and existing results that we will use throughout the paper. Additionally, we establish a special relationship between star compactifications and the Alexandroff compactification. In the third Section we show that the prime spectrum of a von Neumann ring and the prime spectrum of its ring of idempotent elements are homeomorphic. Furthermore, we see that a von Neumann ring is spectrally compact if and only if it has identity. Thus we generalize a well known result of Boolean rings. In the fourth Section we study some relationships between the process of compactification of a von Neumann ring of non-zero characteristic and the process of adjunction of identity. These relationships allows us to identify the compactifications of certain von Neumann rings. Finally we use $p-$rings to illustrate some of the mentioned results.

2. Some Notions and Notations

In the present section we recall without details some notions and facts that we use throughout the paper.

2.1. Spectrally Compact Rings

We say that a commutative ring A is *spectrally compact* if its prime spectrum $Spec(A)$ is a compact topological space. We use this terminology to avoid confusions with compact topological rings. The prime spectrum of the ring A is the set of its prime ideals, endowed with the Zariski topology. In that topology the basic open sets are the sets given by

$$D(a) := \{I : I \text{ is a prime ideal of } A \text{ and } a \notin I\},$$

defined for each $a \in A$. It is known that these basic open sets are compact. If the ring A has identity then $D(1) = Spec(A)$, therefore its prime spectrum is compact. The reciprocal of this statement is false, see that $2\mathbb{Z}$ is a ring without identity and it is spectrally compact because $Spec(2\mathbb{Z}) = D(2)$. On the other hand, we know that a Boolean ring without identity is not spectrally compact. The interested reader can obtain more information in [1] and [5].

2.2. Adjunction of Identity

We recall the Dorroh's mechanism of adjunction of identity presented in [7]. This is a standard procedure to include in a natural way, the ring A of characteristic $n \neq 0$, into a ring with identity and of characteristic n. We consider the set $U_n(A) = A \times \mathbb{Z}_n$ and endow it with the following operations:

$$(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta) \tag{1}$$

$$(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta). \tag{2}$$

If we identify the ring A with $A_0 = A \times \{0\}$ then $U_n(A)$ is a ring with identity $(0, 1)$ that contains the ring A as ideal. This construction satisfies the following universal property.

Proposition 1. *For each unitary ring B of characteristic n and for each ring homomorphism $h : A \to B$ there exists a unique unitary ring homomorphism $\widetilde{h} : U(A) \to B$ such that $\widetilde{h} \circ i_A = h$, where $i_A : A \to U(A) : i_A(a) \mapsto (a, 0)$.*

Proof. It is enough to define $\widetilde{h}(a, \alpha) = h(a) + \alpha 1$, where 1 is the identity of B. ☑

2.3. Compactifications by Finite Points

We mention two of the most known procedures for the compactification of a topological space by finite points.

2.3.1. Alexandroff Compactification

Let (X, τ) be a non-compact topological space and let X^* be the set $X \cup \{\omega\}$, where ω is a point not belonging to X.

Theorem 2. *If $\tau^* = \tau \cup \big\{(X \smallsetminus K) \cup \{\omega\} : K \text{ is a closed-compact set of } X\big\}$, then (X^*, τ^*) is a compactification of (X, τ) by one point.*

This compactification is called the *Alexandroff compactification* of (X, τ) and is the finest of the compactifications of (X, τ) by one point.

Theorem 3. *If (X, τ) is a Hausdorff, locally compact space then (X^*, τ^*) is a Hausdorff space.*

In the context of Hausdorff spaces we mention *the* compactification by one point, because the Alexandroff compactification is the unique compactification of (X, τ) by one point that is a Hausdorff space. See for example [15] for the proofs of these results.

2.3.2. Star Compactifications

We present the definition of *star topology* of [16] and the necessary and sufficient conditions for that topology to give a compactification of the original space. We consider (X, τ) a non-compact topological space, $m \in \mathbb{N}$, $X^\sharp = X \cup \{\omega_1, \dots, \omega_m\}$ where $\omega_1, \dots, \omega_m$ are m different points not belonging to X.

Proposition 4. *If $W_1, \dots, W_m$ are open subsets of X then*

$$\mathcal{B} = \tau \cup \big\{(W_i \smallsetminus K) \cup \{\omega_i\} : K \text{ is a closed-compact set of } X, i \in \{1, \dots, m\}\big\}$$

is a base for a topology μ on $X^\sharp$, which is called the star topology *associated to $W_1, \dots, W_m$.*

Proposition 5. *The space $(X^\sharp, \mu)$ is a compactification of (X, τ) if and only if*

i) the set $X \smallsetminus \bigcup_{i=1}^m W_i$ is compact, and

ii) for each $i \in \{1, \dots, m\}$ and for each closed-compact subset K of X, $W_i \nsubseteq K$.

The proofs of these facts can be read in [16]. Observe that the second condition implies that $W_i \neq \varnothing$, for each $i \in \{1, \dots, m\}$. Furthermore the Alexandroff compactification of a topological space is the unique star compactification by one point of that space, when we take $W_1 = X$. If X is a Hausdorff, locally compact space then each one of its Hausdorff compactifications by m points coincides precisely with a star compactification, where the m open associated sets are pairwise disjoint.

The following result describes a special case of star compactifications.

Theorem 6. *If (X_i, τ_i), $i = 1, \dots, m$, are m non-compact topological spaces, then the sum of its Alexandroff compactifications (X_i^*, τ_i^*) is an $m-$points star compactification of $\coprod_{i=1}^m X_i$.*

Proof. Let $\omega_1, \dots, \omega_m$ be distinct points not belonging to $\coprod_{i=1}^m X_i$ and such that $\omega_i \notin X_i$ for each i. X_i^* is the set $X_i \cup \{\omega_i\}$ endowed with the topology

$$\tau_i^* = \tau_i \cup \big\{(X_i \smallsetminus K_i) \cup \{\omega_i\} : K_i \text{ is a closed-compact set of } X_i\big\},$$

for each $i = 1, \dots, m$. A base for the topology η of $\coprod_{i=1}^m X_i^*$ is

$$\begin{aligned}&\beta_\eta = \big\{A_i \times \{i\} : A_i \in \tau_i, i = 1, \dots, m\big\} \cup \\ &\big\{\big[(X_i \smallsetminus K_i) \cup \{\omega_i\}\big] \times \{i\} : K_i \text{ is a closed-compact set of } X_i, i = 1, \dots, m\big\}.\end{aligned}$$

Notice that $\big[(X_i \smallsetminus K_i) \cup \{\omega_i\}\big] \times \{i\} = \big[(X_i \smallsetminus K_i\big) \times \{i\}\big] \cup \big\{(\omega_i, i)\big\}$.

A base for the topology γ of $\coprod_{i=1}^m X_i$ is $\beta_\gamma = \big\{A_i \times \{i\} : A_i \in \tau_i, i = 1, \dots, m\big\}$.

Let us consider the star compactification of $\coprod_{i=1}^m X_i$ by m points ω_i, $i = 1, \dots, m$ with associated open sets $W_i = X_i \times \{i\}$, $i = 1, \dots, m$. We denote it by $\big(\coprod_{i=1}^m X_i\big)^\sharp$ and μ its topology. It is very easy to check that $W_i; i = 1, \dots, m$ so defined satisfy the conditions of Proposition 5 for $\Big(\big(\coprod_{i=1}^m X_i\big)^\sharp, \mu\Big)$ to be a star compactification of $\big(\coprod_{i=1}^m X_i, \gamma\big)$ by m points. A base for the topology μ is

$$\begin{aligned}&\beta_\mu = \big\{A_i \times \{i\} : A_i \in \tau_i, i = 1, \dots, m\big\} \cup \\ &\big\{\big[(X_i \smallsetminus K_i) \times \{i\}\big] \cup \{\omega_i\} : K_i \text{ is a closed-compact set of } X_i, i = 1, \dots, m\big\}.\end{aligned}$$

Consider the function $f : \Big(\big(\coprod_{i=1}^m X_i\big)^\sharp, \mu\Big) \to \big(\coprod_{i=1}^m X_i^*, \eta\big)$ defined by $f(\omega_i) = (\omega_i, i)$ and $f(x_i, i) = (x_i, i)$ for each $x_i \in X_i$ and each $i = 1, \dots, m$. We can observe that f is bijective. Furthermore, from the description of the basis of the topologies μ and η, it follows that f is the required homeomorphism. ☑

3. Homeomorphic Spectra and Spectral Compactness

In the study of $p-$rings it has been usual to involve Boolean rings (see for example [11] and [19]). We will consider the ring of idempotent elements of a von Neumann ring.

For each commutative ring A, we denote by $E(A)$ the set of idempotent elements of A. Since $E(A)$ is not stable under addition, for $b, c \in E(A)$ we define $b \oplus c = (b-c)^2$ so, $\big(E(A), \oplus, \bullet\big)$ is a Boolean ring. In this section we show that the prime spectrum of a von Neumann ring is homeomorphic to the prime spectrum of its ring of idempotent elements.

Proposition 7. *If A is a von Neumann ring and P is a prime ideal of A, then $P \cap E(A)$ is a prime ideal of $E(A)$.*

Proof. It is enough to show that $P \cap E(A)$ is a proper subset of $E(A)$, because the other details can be easily revised.

Let a be an element of $A - P$. Suppose $P \cap E(A) = E(A)$. Let b be the element of A such that $a = a^2 b$. Clearly $ab \in E(A)$, then $ab \in P$; but this is a contradiction. ☑

Proposition 8. *If A is a von Neumann ring, then*

$$\begin{aligned} f : Spec(A) &\to Spec\big(E(A)\big) \\ P &\mapsto P \cap E(A) \end{aligned}$$

is a continuous, open and one to one function.

Proof. For continuity consider e an element of $E(A)$. It is easy to see that $f^{-1}\big(D(e)\big) = D(e)$.

On the other hand, let a be an element of A and let b be the corresponding element of A such that $a = a^2 b$. As ab is an idempotent element of A and $f\big(D(a)\big) = D(ab)$, then f is open.

Finally, let P, Q be prime ideals of A such that $P \cap E(A) = Q \cap E(A)$. For each $a \in P$ there exists $b \in A$ such that $a = a^2 b$. It is clear that $ab \in P \cap E(A)$, so $ab \in Q \cap E(A)$ and $ab \in Q$. Then $a = a(ab) \in Q$. Therefore $P \subseteq Q$. In a similar way it can be shown that $Q \subseteq P$. ☑

For each subset X of the ring A we denote by $\langle X \rangle_A$ (or simply $\langle X \rangle$ if there is no confusion) the ideal generated by X in A. To prove that the function f in the previous proposition is onto, we need the two following facts.

Lemma 9. *Let B be a Boolean ring. If $b_1, \dots, b_n$ are elements of B then there exist $x_1, \dots, x_n$, elements of B such that*

i) $x_i x_j = 0 \quad \text{if} \quad i \neq j$, and *ii)* $\langle b_1, \dots, b_n \rangle = \langle x_1, \dots, x_n \rangle$.

Proof. Take $x_1 = b_1$, so $\langle x_1 \rangle = \langle b_1 \rangle$.

Suppose we have built $x_1, \dots, x_k$ such that $x_i x_j = 0$ if $i \neq j$ and $\langle b_1, \dots, b_k \rangle = \langle x_1, \dots, x_k \rangle$.

Let $x_{k+1} = b_{k+1} + b_{k+1}(x_1 + x_2 + \cdots + x_k)$. For $i = 1, \dots, k$,

$$x_{k+1}x_i = b_{k+1}x_i + b_{k+1}x_i^2 + b_{n+1}x_i(x_1 + x_2 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_k) = 0.$$

On the other hand, it is clear that $\big\langle \{x_1, x_2, \dots, x_{k+1}\} \big\rangle \subseteq \big\langle \{b_1, b_2, \dots, b_{k+1}\} \big\rangle$ and furthermore

$$b_{k+1} = x_{k+1} - b_{k+1}(x_1 + x_2 + \cdots + x_k) \in \big\langle \{x_1, x_2, \dots, x_{k+1}\} \big\rangle;$$

then $\big\langle \{x_1, x_2, \dots, x_{k+1}\} \big\rangle = \big\langle \{b_1, b_2, \dots, b_{k+1}\} \big\rangle$.

Therefore the set $\{x_1, \dots, x_{n+1}\}$ is such that $\big\langle b_1, \dots, b_{n+1} \big\rangle = \big\langle x_1, \dots, x_{n+1} \big\rangle$ and $x_i x_j = 0$ if $i \neq j$. ☑

Proposition 10. *If A is a von Neumann ring and J is a proper ideal of $E(A)$ then $\langle J \rangle_A$ is a proper ideal of A.*

Proof. Let $e \in E(A) - J$. Suppose that $e \in \langle J \rangle_A$, then there exist $b_1, \dots, b_m \in J$ and $a_1, \dots, a_m \in A$ such that $e = a_1 b_1 + \cdots + a_m b_m$. By the previous lemma there exist $x_1, \dots, x_m \in J$ such that $x_i x_j = 0$ if $i \neq j$ and $\langle b_1, \dots, b_m \rangle_J = \langle x_1, \dots, x_m \rangle_J$. Thus $b_i = e_{i1}x_1 + \cdots + e_{im}x_m$, for some $e_{i1}, \dots, e_{im} \in J$, so

$$\begin{aligned} e &= a_1(e_{11}x_1 + \cdots + e_{1m}x_m) + \cdots + a_m(e_{m1}x_1 + \cdots + e_{mm}x_m) \\ &= (a_1e_{11} + \cdots + a_m e_{m1})x_1 + \cdots + (a_1 e_{1m} + \cdots + a_m e_{mm})x_m \\ &= c_1 x_1 + \cdots + c_m x_m, \end{aligned}$$

with $c_1, \dots, c_m \in A$. Then $ex_i = c_i x_i$ for all i and

$$e = ex_1 + \cdots + ex_m \in J,$$

but this is absurd. Therefore $e \notin \langle J \rangle_A$. ☑

Notice that there is no confusion with the additions in $E(A)$ and A, because $u \oplus v = u + v$ when $uv = 0$.

Corollary 11. *If A is a von Neumann ring, then the function*

$$\begin{aligned} Spec(A) &\to Spec\big(E(A)\big) \\ P &\mapsto P \cap E(A) \end{aligned}$$

is onto.

Proof. Let J be a prime ideal of $E(A)$. By the previous proposition $\langle J\rangle_A$ is a proper ideal of A. Let $a \in A \setminus \langle J\rangle_A$. Since A is a von Neumann ring, then $\{a^n : n \in \mathbb{Z}^+\}$ is a multiplicative set, disjoint from $\langle J\rangle_A$. By the Prime Ideal Theorem, there exists a prime ideal P of A such that $\langle J\rangle_A \subseteq P$. Furthermore $J \subseteq P \cap E(A)$. As the prime ideals of $E(A)$ are maximal, then $J = P \cap E(A)$. ☑

We summarize the previous results in the following theorem.

Theorem 12. *If A is a von Neumann ring, then $Spec(A)$ and $Spec\big(E(A)\big)$ are canonically homeomorphic under the map $P \mapsto P \cap E(A)$.*

This result was presented in [17], but the authors, in their proof, use the hypothesis that the ring A has identity.

Corollary 13. *Let A be a von Neumann ring. Then, A is spectrally compact if and only if $E(A)$ is spectrally compact.*

Although the following corollaries mention results already known (see [5]), they have been obtained in this work in a different way.

Corollary 14. *The prime spectrum of a von Neumann ring is a Hausdorff space.*

Corollary 15. *The prime ideals of a von Neumann ring are maximal.*

In the remainder of this section we characterize the spectral compactness of von Neumann rings.

Proposition 16. *If A is a von Neumann ring, then A has identity if and only if $E(A)$ has identity.*

Proof. Suppose that 1 is the identity of A. As 1 is an idempotent element, then 1 is the identity of $E(A)$.

On the other hand, let e be the identity of $E(A)$. For $a \in A$ there exists $b \in A$ such that $a = a^2b$. Thus $a = a(ab) = a\big((ab)e\big) = (a^2b)e = ae$. Then e is the identity of A. ☑

The following corollary is immediate.

Corollary 17. *Let A be a von Neumann ring. Then, A is spectrally compact if and only if A has identity.*

Example 18. Let A be a von Neumann ring. The ring $A^{(\mathbb{N})}$ of the sequences with finitely many non-zero elements of A is a von Neumann ring without identity, thus it is a ring that is not spectrally compact.

4. On Spectral Compactification of von Neumann Rings

Let A be a von Neumann ring of characteristic $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where p_i are different prime numbers. As $\pi : U_n(A) \to \mathbb{Z}_n$ defined by $\pi(a, \alpha) = \alpha$ is a homomorphism of unitary rings, then the prime ideals of $U_n(A)$ that contain A_0 are precisely the ideals of the form $A \times p_i\mathbb{Z}_n$, for $i = 1, \ldots, m$, one for each prime divisor of n. We denote by $D(A_0)$ the subspace of $Spec\big(U_n(A)\big)$ whose elements are the prime ideals of $U_n(A)$ that do not contain A_0.

Theorem 19. *The function $\psi_n : Spec(A) \to D(A_0)$ defined by*

$$\psi_n(I) = \big\{(a, \alpha) \in U_n(A) : (a, \alpha)A_0 \subseteq I \times \{0\}\big\},$$

is a homeomorphism and its inverse is

$$\begin{aligned} \varphi : D(A_0) &\to Spec(A) \\ J &\mapsto J \cap A_0. \end{aligned}$$

Proof. Since A_0 is an ideal of $U_n(A)$, it is easily seen that ψ_n is well defined and $\psi_n^{-1} = \varphi$. On the other hand, if $(a, \alpha) \in U_n(A)$ then $\psi_n^{-1}\big(D(a, \alpha)\big) = \bigcup_{x \in A} D(ax + \alpha x)$. Thus, ψ_n is continuous. Furthermore if $b \in A$, it is clear that $\varphi^{-1}\big(D(b)\big) = D(b) \cap D(A_0)$. Thus φ is continuous. ☑

Because of this result we can say that the prime spectrum of A is a subspace of the prime spectrum of $U_n(A)$.

***Remark* 20.** From the previous observations we can conclude that if A does not have identity, then $Spec\big(U_n(A)\big)$ contains a compactification of $Spec(A)$ by at most m points. In particular, if $n = p^\alpha$ then $Spec\big(U_n(A)\big)$ is a compactification of $Spec(A)$ by exactly one point. If B is a Boolean ring without unity then $Spec\big(U_2(B)\big)$ is a compactification by one point of $Spec(B)$. In [2] it is showed that in this case, $Spec\big(U_2(B)\big)$ is precisely the Alexandroff compactification of $Spec(B)$.

Theorem 21. *If A is a von Neumann ring of characteristic $n \neq 0$, then n is square free.*

Proof. Suppose that $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ and for some $i \in \{1, \ldots, m\}$, $\alpha_i > 1$. If we call μ the maximum of the set $\{\alpha_1, \ldots, \alpha_m\}$, clearly $\mu > 1$. There exists $a \in A$ such that $p_1 \cdots p_m a \neq 0$, but $(p_1 \cdots p_m a)^\mu = 0$ because n divides $(p_1 \cdots p_m)^\mu$. Thus, $p_1 \cdots p_m a$ is a non-zero nilpotent element of A, but this contradicts that A is a von Neumann ring. ☑

Proposition 22. *If A is a von Neumann ring of characteristic $n = p$, where p is a prime number, then $Spec\big(U_n(A)\big)$ and $Spec\big(U_2\big(E(A)\big)\big)$ are homeomorphic.*

Proof. Let $\eta : Spec\big(U_n(A)\big) \to Spec\big(U_2\big(E(A)\big)\big)$ be the function defined by $\eta(A \times p\mathbb{Z}_n) = E(A) \times \{0\}$ and $\eta\big(\psi_n(P)\big) = \psi_2\big(P \cap E(A)\big)$, where $P \in Spec(A)$.

Clearly η is bijective because if η is restricted to $Spec\big(U_n(A)\big) \smallsetminus \{A \times p\mathbb{Z}_n\}$, it is the homeomorphism $\psi_2 \circ f \circ \varphi$ onto $Spec\big(U_2\big(E(A)\big)\big) \smallsetminus \{E(A) \times \{0\}\}$; where f is as in Proposition 8 and the homeomorphisms ψ_2 and φ are as in Theorem 19.

Since $Spec\big(U_n(A)\big)$ is a compact space and $Spec\big(U_2\big(E(A)\big)\big)$ is a Hausdorff space, we only have to verify the continuity of η. For this purpose it is enough to consider the basic open sets that contain $E(A) \times \{0\}$.

Let $(a, \alpha) \in U_2\big(E(A)\big)$ such that $E(A) \times \{0\} \in D(a, \alpha)$, so $(a, \alpha) \notin E(A) \times \{0\}$. Then $\alpha = 1$.

We see that $\eta^{-1}\big(D(a, 1)\big)$ is an open set in $Spec\big(U_n(A)\big)$ showing that all its points are interior points.

(1) Since $A \times p\mathbb{Z}_n \in \eta^{-1}\big(D(a, 1)\big)$ and $(a, -1) \notin A \times p\mathbb{Z}_n$, then $A \times p\mathbb{Z}_n \in D(a, -1)$.

   Let $J \in Spec(A)$ such that $\psi_n(J) \in D(a, -1)$, so $(a, -1) \notin \psi_n(J)$ and there exists $x \in A \smallsetminus J$ such that $ax - x \notin J$. As A is a von Neumann ring then there exists $y \in A \smallsetminus J$ such that $x = x^2 y$. Take $e = xy$ an idempotent element that does not belong to J. As $ae - e = axy - xy = (ax - x)y \notin J$ and $e \oplus ae = (e - ae)^2 = e - ae$ then $e \oplus ae \notin J \cap E(A)$. Thus $(a, 1) \notin \psi_2\big(J \cap E(A)\big)$ and $\psi_2\big(J \cap E(A)\big) = \eta\big(\psi_n(J)\big) \in D(a, 1)$. Therefore $\psi_n(J) \in \eta^{-1}\big(D(a, 1)\big)$.

(2) If $\psi_n(I) \in \eta^{-1}\big(D(a, 1)\big)$ then $\psi_2\big(I \cap E(A)\big) \in D(a, 1)$, that is $(a, 1) \notin \psi_2\big(I \cap E(A)\big)$. Thus there exists $e \in E(A)$ such that $ae \oplus e \notin I \cap E(A)$. As $ae \oplus e = (ae - e)^2 = e - ae$ then $e - ae \notin I$, so $(e - ae, 0) \notin \psi_n(I)$ and $\psi_n(I) \in D(e - ae, 0)$. Now we have to see that $D(e - ae, 0) \subseteq \eta^{-1}\big(D(a, 1)\big)$.

   Let $J \in Spec(A)$ such that $\psi_n(J) \in D(e - ae, 0)$; then $(e - ae, 0) \notin \psi_n(J)$, thus $e - ae \notin J$. So $ae \oplus e \notin J \cap E(A)$ and $(a, 1) \notin \psi_2\big(J \cap E(A)\big) = \eta\big(\psi_n(J)\big)$. Thus $\eta\big(\psi_n(J)\big) \in D(a, 1)$ and $\psi_n(J) \in \eta^{-1}\big(D(a, 1)\big)$. ☑

We recall that if R is a commutative, unitary ring then $Spec(R)$ is homeomorphic to $Spec\big(R/N(R)\big)$, where $N(R)$ is the nilradical of R. Furthermore $Spec(R)$ is a Hausdorff space if and only if $R/N(R)$ is a von Neumann ring (see for example [5]). With this observation we obtain the following corollary:

Corollary 23. *If A is a von Neumann ring of characteristic $n = p$, where p is a prime number, then $U_n(A)$ is a von Neumann ring.*

Proof. By the previous proposition $Spec\big(U_n(A)\big)$ is a Hausdorff space then, $U_n(A)/N\big(U_n(A)\big)$ is a von Neumann ring. Take $(a, \beta) \in N\big(U_n(A)\big)$. So $\beta^k = 0$

for some positive integer k, but this is only possible if $\beta = 0$. Thus $(a, \beta) = (0, 0)$ because $N(A) = 0$. Then $N\big(U_n(A)\big) = 0$ and $U_n(A)$ is a von Neumann ring. ☑

Proposition 24. *If A is a von Neumann ring without identity and of characteristic $n = p$, where p is a prime number, then $Spec\big(U_n(A)\big)$ is the Alexandroff compactification of $Spec(A)$.*

Proof. By Proposition 22 we have that $Spec\big(U_n(A)\big)$ is a Hausdorff space. On the other hand, $Spec\big(U_n(A)\big)$ is an one-point compactification of $Spec(A)$. Then $Spec\big(U_n(A)\big)$ is the Alexandroff compactification of $Spec(A)$. ☑

The *A-spectral spaces* were defined in [6]. They are topological spaces such that its Alexandroff compactification is an spectral space. The spectral spaces are precisely the prime spectra of commutative unitary rings (see [12]). Therefore by the earlier proposition we obtain the following fact.

Corollary 25. *If A is a von Neumann ring without identity and of prime characteristic, then $Spec(A)$ is an A-spectral space.*

Proposition 26. *If A is a von Neumann ring and J is an ideal of A, then J is a von Neumann ring.*

Proof. Let x be an element of J. There exists $y \in A$ such that $x = x^2y$. Take $e = xy$ an idempotent element of J. As $x = xe = xe^2 = xxyxy = x^2(xy^2)$ then $x = x^2z$, where $z = xy^2 \in J$. ☑

Every commutative ring of non-zero characteristic with at least two prime divisors can be decomposed as a product of rings with special characteristics.

Lemma 27. *If A is a commutative ring of characteristic $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where the $p_i's$ are distinct primes, then there exist commutative rings $B_1, \dots, B_m$ such that $A \cong \prod_{i=1}^m B_i$ and $\operatorname{char}(B_i) = p_i^{\alpha_i}$, for each $i = 1, \dots, m$. In addition this decomposition is unique up to isomorphism.*

Proof. We proceed by induction defining $B_i = \{x \in A : p_i^{\alpha_i}x = 0\}$, for each i. ☑

We will call this type of decomposition for a commutative ring of non-zero characteristic *the characteristic decomposition of the ring.*

Corollary 28. *If A is a commutative ring of characteristic $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where the $p_i's$ are distinct primes and $\prod_{i=1}^m A_i$ is the characteristic decomposition of A, then A_i is an ideal of A, for each $i = 1, \dots, m$.*

Proposition 29. *If A is a commutative ring of characteristic $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where the $p_i's$ are distinct primes, and $\prod_{i=1}^m A_i$ is the characteristic decomposition of A, then $\prod_{i=1}^m U_{p_i^{\alpha_i}}(A_i)$ is the characteristic decomposition of $U_n(A)$.*

Proof. If we call $R = \prod_{i=1}^{m} U_{p_i^{\alpha_i}}(A_i)$ then $\rho : U_n(A) \to R$ defined by $\rho\big((a_i)_{i=1}^m, \alpha\big) = \big((a_i, [\alpha]_{p_i^{\alpha_i}})\big)_{i=1}^m$ is an onto homomorphism of unitary rings. We see that ρ is also injective.

Take $\big((a_i)_{i=1}^m, \alpha\big) \in \ker \rho$, that is, $\big((a_i, [\alpha]_{p_i^{\alpha_i}})\big)_{i=1}^m = \big((0,0)\big)_{i=1}^m$. So $a_i = 0$ for each i and $[\alpha]_{p_i^{\alpha_i}} = 0$ for each i. Thus α is a multiple of $p_i^{\alpha_i}$, for each i. Then α is a multiple of n, so $\alpha = 0$. ☑

Proposition 30. *If A is a von Neumann ring of characteristic $n \neq 0$, then $U_n(A)$ is a von Neumann ring.*

Proof. By Theorem 21, suppose that $n = p_1 \cdots p_m$, a product of different primes, and let $\prod_{i=1}^m A_i$ be the characteristic decomposition of A. Because A_i is an ideal of A then, by Proposition 26, A_i is a von Neumann ring for each $i = 1, \dots, m$. As $U_n(A) \cong \prod_{i=1}^m U_{p_i}(A_i)$ where each $U_{p_i}(A_i)$ is a von Neumann ring, then $U_n(A)$ is a von Neumann ring. ☑

As a consequence of the topological result presented in Theorem 6, we obtain the following fact related to the compactification of certain von Neumann rings without identity and of non-zero characteristic.

Theorem 31. *If A is a von Neumann ring of characteristic $n = p_1 \cdots p_m$ and every factor in the characteristic decomposition of A does not have identity, then $Spec\big(U_n(A)\big)$ is a star compactification of $Spec(A)$ by m points.*

Proof. Let $\prod_{i=1}^m A_i$ be the characteristic decomposition of A, where each A_i has no identity. As each A_i is an ideal of A, then each A_i is a von Neumann ring without identity. Because $U_n(A) \cong \prod_{i=1}^m U_{p_i}(A_i)$ then $Spec\big(U_n(A)\big) \approx \coprod_{i=1}^m Spec\big(U_{p_i}(A_i)\big)$, where each $Spec\big(U_{p_i}(A_i)\big)$ is the Alexandroff compactification of $Spec(A_i)$. Then by Theorem 6, $Spec\big(U_n(A)\big)$ is a star compactification of $Spec(A)$ by m points. ☑

5. Some Examples

Let p be a prime number and let A be a $p-$ring. By Corollary 23 we know that $U_p(A)$ is a von Neumann ring. The following proposition guarantees that in particular it is a $p-$ring.

Proposition 32. *The ring $U_p(A)$ is a $p-$ring.*

Proof. It is clear that $U_p(A)$ is of characteristic p. Let $(a, \alpha) \in U_p(A)$.

$(a,\alpha)^p = \left(a^p + \sum_{k=1}^{p-1} \binom{p}{k} \alpha^k a^{p-k}, \alpha^p\right) = (a, \alpha)$ because $\binom{p}{k}$ is a multiple of p, for each $k = 1, \dots, p-1$. ☑

As a consequence of Proposition 24 we know that if A is a $p-$ring without identity, then the prime spectrum of $U_p(A)$ is precisely the Alexandroff compactification of $Spec(A)$. Therefore $Spec(A)$ is an A-spectral space.

Example 33. The Alexandroff compactification of the prime spectrum of the non-spectrally compact $p-$ring $\mathbb{Z}_p^{(\mathbb{N})}$ is the prime spectrum of $U_p\Big(\mathbb{Z}_p^{(\mathbb{N})}\Big)$. Through the universal property of U_p (see Proposition 1), we can see that $U_p\Big(\mathbb{Z}_p^{(\mathbb{N})}\Big)$ is the subring of almost constant sequences of $\mathbb{Z}_p^{\mathbb{N}}$. Indeed let h be the inclusion homomorphism from $\mathbb{Z}_p^{(\mathbb{N})}$ to $\mathbb{Z}_p^{\mathbb{N}}$, $i : \mathbb{Z}_p^{(\mathbb{N})} \to U_p\Big(\mathbb{Z}_p^{(\mathbb{N})}\Big)$ defined by $i(a_i) = \big((a_i), 0\big)$, so $\widetilde{h} : U_p\Big(\mathbb{Z}_p^{(\mathbb{N})}\Big) \to \mathbb{Z}_p^{\mathbb{N}}$ is defined by $\widetilde{h}\big((a_i), \alpha\big) = (a_i) + \alpha(1) = (a_i + \alpha.1) = (b_i)$. As $a_i = 0$ for almost all i, then $b_i = \alpha$, for almost all i. The image of $\widetilde{h}$ is the subring of almost constant sequences of $\mathbb{Z}_p^{\mathbb{N}}$.

Example 34. If we consider the ring $A = \prod_{i=1}^{m} A_i$, where $A_i = \mathbb{Z}_{p_i}^{(\mathbb{N})}$ and the p_i are different prime numbers for $i = 1, \ldots, m$ then the prime spectrum of $U_n(A)$ is a star compactification of the spectrum of A by exactly m points.

Question 35. Is it possible to characterize algebraically all the commutative rings that are spectrally compact if and only if they have identity?

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