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SYMMETRIC PERTURBATION OF A SELF-ADJOINT OPERATOR

by

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SUMMARY

We give a proof, which is valid in both real and complex Hilbert spaces, of the following result: If A is self-adjoint, B is symmetric $D(B) \supset D(A)$, $||Bx|| < a\,||x|| + ||Ax||$ for $x \in D(A)$, then $A+bB$ is self-adjoint for $0<b<\frac{1}{2}$.

Introduction. **In a complex Hilbert Space, it is known that the operator $A+B$ is self-adjoint, if A is self-adjoint and B symmetric with $D(B) \supset D(A)$ and $||Bx|| \leq a\,||x|| + b\,||Ax||$ for all $x \in D(A)$, where $0 < b < 1$. We give a proof which is valid in both real and complex Hilbert Spaces. The proof of the result in a complex Hilbert Space is given in [1]. Notice that A and B are not necessarily bounded, $A=A^*$ and $B \subset B^*$.**

LEMMA 1. Let A be a closed operator in a Hilbert Space H and B an operator with $D(B) \supset D(A)$ and $||Bx|| \leq a.||x|| + ||Ax||$. Let b and k be real numbers such that $0< b < 1$ and $k > \frac{2ab}{1-b}$. Then for $\forall z \in D(A) \exists z' \in D(A) \ni (Az, Ax) +$

$$k^2(z, x) = ((A + bB)z', Ax) + k^2(z', x) \quad \text{for all } x \in D(A).$$

Proof: Let $H_1 = D(A)$ with the inner product

$$(x, y)_1 = (Ax, Ay) + k^2(x, y).$$

H_1 is a Hilbert Space. Consider the linear functional f_1 on H_1 given by

$$f_1(x) = -(Bz, Ax)\, b,$$

$$|f_1(x)| \leq b\,||Ax||\,||Bz|| \leq b\,||Bz||\,||x||_1 < \frac{1+b}{2}\,||x||_1\,||z||_1.$$

By the representation theorem $\exists z_1 \in H_1$ such that $(z_1, x)_1 = f_1(x) = -b(Bz, Ax)$ and $||z_1||_1 = ||f_1|| < \frac{1+b}{2}\,||z||_1$. Define the sequence $\{z_n\} \subset H_1$ inductively by

$z_o = z$,

$$(z_1, x)_1 = -b(Bz, Ax),$$

$$(z_{n+1}, x)_1 = -b(Bz_n, Ax).$$

Then we have

$$(z_o, x)_1 = \sum_{s=o}^{n} [((A + bB)z_s, Ax) + k^2(z_s, x)] + (z_{n+1}, x)_1.$$

The inequality

$$\sum_{s=1}^{n} ||z_s||_1 < \left(\sum_{1}^{n} \left(\frac{1+b}{2}\right)^s\right) ||z_o||_1$$

shows that ΣAz_s and Σz_s are absolutely convergent. Let $\sum_{o}^{\infty} z_s = z'$. The convergence of $\Sigma||Az_s||$ implies the convergence of $\Sigma||(A+bB)z_s||$. Since $A+bB$ is closed, $\Sigma(A+bB)z_s \to (A+bB)z'$.

LEMMA 2. Let S be closed and injective, $D(T) \supset D(S)$ and $||Tx|| \leq b\,||Sx||$ for all $x \in D(S)$ with $0 < b < 1$. Then $R(S+T) = H$ if $R(S) = H$.

Proof. Let $z_o \epsilon H$, $Sx_o = z_o$. Define the sequence $\{x_n\}$ by $Sx_{n+1} = -Tx_n$, $n = 0, 1, 2, \ldots$. Then $\sum_{k=o}^{m} ||Sx_k|| \leq ||z_o|| \sum_{k=o}^{m} b^k$ implies that $\sum_{o}^{m} ||Sx_k||$ converges. Hence ΣSx_k converges. Since $R(S) = H$, S^{-1} is bounded and Σx_k converges to say x. $\Sigma ||(S+T)x_k||$ is bounded. Since $S+T$ is closed se have $\Sigma (S+T)x_k \to (S+T)x$. Thus $(S+T)x = z_o$.

LEMMA 3. Let A be self-adjoint and $Bx \leq a||x|| + ||Ax||$. Then $R[k^2 + A^2 + bBA] = H$ for $k > \frac{ab}{1-2b}$, $0 < b < \frac{1}{2}$.

Proof. For $0 < b < \frac{1}{2}$, we have

$$b||BAx|| \leq (2 + \frac{a}{2k})b \, || A^2x + k^2x ||$$

$$\leq \beta \, || A^2x + k^2x || \quad \text{for some } \beta, \; 0 < \beta < 1 .$$

Since $R[k^2 + A^2] = H$, we have, using Lemma 2, $R[k^2 + A^2 + bBA] = H$.

THEOREM. Let A be self-adjoint, B symmetric, $D(B) \supset D(A)$, $||Bx|| \leq a||x|| + ||Ax||$ for $x \epsilon D(A)$. Then $A + bB$ is self-adjoint for $0 < b < \frac{1}{2}$.

Proof. Since $A+bB$ is closed and symmetric, it suffices to prove that

$$D(A + bB)^* \subset D(A + bB).$$

Let $y \epsilon D(A+bB)^*$. Then we have

$$|((A+bB)^* y, Ax) + k^2(y,x)|$$

$$\leq (||(A+bB)^* y|| + ||y||)\,||x||_1 .$$

Thus the linear functional g on H_1 given by $g(x) = ((A+bB)^* y, Ax) + k^2(y,x)$ is bounded. Thus $\exists\, z \epsilon D(A)$ such that

$$g(x) = ((A + bB)\, z', Ax) + k^2 (z', x) \quad \text{for all } x \in D(A).$$

Thus we have, since $A+bB$ is symmetric,

$$(y - z', ((A + bB)A + k^2)\, x) = 0 \quad \text{for all } x \in D(A^2).$$

Since $R[k^2 + (A + bB)A] = H$, we have $y = z'$. Hence $D(A + bB)^* = D(A + bB)$.

COROLLARY. The Theorem is true for all b with $0 < b < 1$.

Proof. Let $A_0 = A$, $A_{n+1} = A_n + \dfrac{\alpha}{2^{n+1}} B$, $n = 0, 1, 2, \ldots m$, where

$$b = \alpha \left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^m} \right] \text{ and } 0 < \alpha < 1.$$

Then

$$\frac{\alpha}{2^{n+1}} \|Bx\| \leq (\|A_n x\| + a_n \|x\|) \frac{\alpha}{2},$$

for some a_n. Induct on n.

REFERENCE

1. KATO, T. *Perturbation Theory for Linear Operators.* Springer-Verlag, New York, Inc., New York, 1966.

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