SOME BOUNDARY-VALUE PROPERTIES OF AN ANALYTIC FUNCTION
IN A BICYLINDER

by

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SUMMARY

A theorem of Fatou is extended to the bicylinder.

Let $D_1, D_2$ be two copies of the unit disc, $D = D_1 \times D_2$, and $\partial D_1 \times \partial D_2 \neq \partial_0 D$ the distinguished boundary of $D$. Let $A$ be the set of all functions $u$, continuous in $\bar{D}$ and holomorphic in $D$; let $A_j (j=1, 2)$ be the set of all functions $u$, continuous in $\bar{D}_j$ and holomorphic in $D_j$.

For $D_1$ we have the following

THEOREM (Fatou). Let $K_1$ be a closed set of Lebesgue measure zero on the boundary of $D_1$. Then there exists a function in $A_1$ which vanishes precisely on $K_1$. (See Hoffman [1], p. 80).

Our purpose is to study the possibility of an extension of this theorem to the bicylinder $D$.

PROPOSITION. If $u \in A$, and $\xi_1 \in \partial D_1$ then the function $z_2 \mapsto u(\xi_1, z_2)$ belong to $A_2$.

Proof. Let $z_{1n}$ be a sequence in $D_1$, converging to $\xi_1$, and let
We have: i) $u_n \in A_2$, ii) $|u_n|$ is uniformly bounded, iii) $u_n \to u(\zeta_1, z_2)$ pointwise. The proposition follows from the Stieltjes-Vitali theorem (cfr. Hörmander [2], Cor. 1.2.6).

Let $K$ be a closed subset of the distinguished boundary of $D$ and write

$$K_{\zeta_1} = \{ \zeta_2 : (\zeta_1, \zeta_2) \in K \} \quad \text{for} \quad \zeta_1 \in \partial D_1,$$

$$K_{\zeta_2} = \{ \zeta_1 : (\zeta_1, \zeta_2) \in K \} \quad \text{for} \quad \zeta_2 \in \partial D_2.$$

If $K_j$ is a measurable set included in $\partial D_j$, we shall denote by $|K_j|$ its linear Lebesgue measure.

**DEFINITION.** Let $K$ be a closed subset of $\partial_0 D$;

a) if for every $\zeta_j \in \partial D_j$ we have either $|K_{\zeta_j}| = 0$ or $|K_{\zeta_j}| = 2\pi$ for $j = 1, 2$, we say that $K$ is a separately Fatou set of $\overline{D}$;

b) if there exists a function $f \in A$ satisfying $f(z) = 0$ if $z \in K$ and $f(z) \neq 0$ if $z \in D\setminus K$, we say that $K$ is a Fatou set of $\overline{D}$, or simply a Fatou set.

If $K$ is a Fatou set, the proposition and known properties of the boundary-values of holomorphic functions of one variable show that it is a separately Fatou set of $\overline{D}$, but the converse is not true as the following example communicated by Professor E. Stein shows:

**Example.** (1) Consider first a function $\phi(z_1, z_2)$ holomorphic in the cartesian product of the half-planes, $Rz_1 < 0$, $Rz_2 < 0$ and continuous in the closure (except at $\infty$). Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ ($x_1 < 0$, $x_2 < 0$) and suppose that $\phi$ vanishes in a segment (or simply in a set of positive Lebesgue measure) of the line $y_2 = ky_1$, $k > 0$ of the distinguished boundary $x_1 = 0, x_2 = 0$. Then the function $\chi(z) = \phi(z, kz)$ holomorphic in a half-plane and continuous in its closure, vanishes on a segment of the boundary, and so vanishes identically.
(2) Now let \( f(\zeta_1, \zeta_2) \) be holomorphic in the bicylinder \(|\zeta_1| < 1, |\zeta_2| < 1\), continuous in its closure, and vanish on a segment of the line \( \theta_2 = k\theta_1 \), of the line \( \theta_2 = k\theta_1 \), of the distinguished boundary \((\zeta_1 = \exp i\theta_1, \zeta_2 = \exp i\theta_2)\) where \( k \) is positive and irrational. Write \( \zeta_1 = \exp z_1, \zeta_2 = \exp z_2 \), and consider the function
\[
\phi(z_1, z_2) = f(\exp z_1, \exp z_2), \quad z_j = x_j + iy_j
\]
which has the properties required in (1). This function vanishes on the whole straight line \( \theta_2 = k\theta_1 \), generated by the segment in the distinguished boundary of \( D \).

In view of the periodicity of \( \phi(iy_1, iy_2) \) and the irrationality of \( k \), the function \( \phi(iy_1, iy_2) \) is zero in a dense set and therefore zero identically. Hence \( f \equiv 0 \).

REFERENCES


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