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SOME BOUNDARY - VALUE PROPERTIES OF AN ANALYTIC FUNCTION IN A BICYLINDER

by

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SUMMARY

A theorem of Fatou is extended to the bycilinder.

Let D_1, D_2 be two copies of the unit disc, $D = D_1 \times D_2$, and $\partial D_1 \times \partial D_2 \neq \partial_0 D$ the distinguished boundary of D. Let A be the set of all functions u, continuous in $\bar{D}$ and holomorphic in D; let $A_j (j=1,2)$ be the set of all functions u, continuous in $\bar{D}_j$ and holomorphic in D_j.

For D_1 we have the following

THEOREM (Fatou). *Let K_1 be a closed set of Lebesgue measure zero on the boundary of D_1. Then there exists a function in A_1 which vanishes precisely on K_1.* (See Hoffman [1], p. 80).

Our purpose is to study the possibility of an extension of this theorem to the bicylinder D.

PROPOSITION. If $u \in A$, and $\xi_1 \in \partial D_1$ then the function $z_2 \mapsto u(\zeta_1, z_2)$ belong to A_2.

Proof. Let z_{1n} be a sequence in D_1, converging to ζ_1, and let

$u_n : z_2 \mapsto u(z_{1n}, z_2)$. We have : i) $u_n \in A_2$, ii) $|u_n|$ is uniformly bounded, iii) $u_n \to u(\zeta_1, z_2)$ pointwise. The proposition follows from the Stieltjes-Vitali theorem (cfr. Hörmander [2], Cor. 1.2.6).

Let K be a closed subset of the distinguished boundary of D and write

$$K_{\zeta_1} = \{\zeta_2 ; (\zeta_1, \zeta_2) \in K\} \qquad \text{for} \qquad \zeta_1 \in \partial D_1 ,$$

$$K_{\zeta_2} = \{\zeta_1 ; (\zeta_1, \zeta_2) \in K\} \qquad \text{for} \qquad \zeta_2 \in \partial D_2 .$$

If K_j is a measurable set included in ∂D_j, we shall denote by $|K_j|$ its linear Lebesgue measure.

DEFINITION. Let K be a closed subset of $\partial_0 D$;

a) if for every $\zeta_j \in \partial D_j$ we have either $|K_{\zeta_j}| = 0$ or $|K_{\zeta_j}| = 2\pi$ for $j = 1, 2,$ we say that K is a separately Fatou set of $\bar{D}$;

b) if there exists a function $f \in A$ satisfying $f(z) = 0$ if $z \in K$ and $f(z) \neq 0$ if $z \in D-K$, we say that K is a Fatou set of $\bar{D}$, or simply a Fatou set .

If K is a Fatou set, the proposition and known properties of the boundary-values of holomorphic functions of one variable show that it is a separately Fatou set of $\bar{D}$, but the converse is not true as the following example communicated by Proffesor E. Stein shows :

Example. (1) Consider first a function $\phi(z_1,z_2)$ holomorphic in the cartesian product of the half-planes, $Rz_1 < 0$, $Rz_2 < 0$ and continuous in the closure (except at ∞). Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ $(x_1 < 0 , x_2 < 0)$ and suppose that ϕ vanishes in a segment (or simply in a set of positive Lebesgue measure) of the line $y_2 = ky_1$, $k > 0$ of the distinguished boundary $x_1 = 0, x_2 = 0$. Then the function $\chi(z) = \phi(z, kz)$ holomorphic in a half-plane and continuous in its closure, vanishes on a segment of the boundary, and so vanishes identically .

(2) Now let $f(\zeta_1, \zeta_2)$ be holomorphic in the bicylinder $|\zeta_1| < 1, |\zeta_2| < 1$, continuous in its closure, and vanish on a segment of the line $\theta_2 = k\theta_1$, of the line $\theta_2 = k\theta_1$, of the distinguished boundary $(\zeta_1 = exp\ i\theta_1, \zeta_2 = exp\ i\theta_2)$ where k is positive and irrational. Write $\zeta_1 = exp\ z_1, \zeta_2 = exp\ z_2$, and consider the function

$$\phi(z_1, z_2) = f(exp\ z_1, exp\ z_2)\ ,\ z_j = x_j + iy_j$$

which has the properties required in (1). This function vanishes on the whole straight line $\theta_2 = k\theta_1$, generated by the segment in the distinguished boundary of D. In view of the periodicity of $\phi(iy_1, iy_2)$ and the irrationality of k, the function $\phi(iy_1, iy_2)$ is zero in a dense set and therefore zero identically. Hence $f \equiv 0$.

REFERENCES

1. *Hoffman, Kenneth, Banach Spaces of Analytic Functions.* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962.

2. *Hörmander, Lars, An Introduction to Complex Analysis in Several Variables.* D. van Nostrand Company, Inc., Princeton, N. J., 1966.

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