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ON THE UNIVALENCE OF QUASI-ISOMETRIC MAPPINGS

by

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Let X and X' be two Banach spaces and let $G \subset B$ be open. John [1] defines a quasi-isometric mapping $f: G \to X'$ to be an open (i.e., maps open sets onto open sets) local homeomorphism for which

$$(U) \qquad \infty > M \geq \limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|}$$

and

$$(L) \qquad 0 < m \leq \liminf_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|}$$

for all points x in G. More precisely, we will call such a mapping f an (m, M)-isometric mapping. The purpose of this paper is to offer a proof of the following

THEOREM. Let $f: B \to E^n$ be an (m,M)-isometric mapping, where B is an open ball in E^n, the n-dimensional Euclidean space. Then f is univalent (i.e., one to one) if $M/m \leq \frac{4 + 2\pi}{4 + \pi} = 1.439\ldots$.

Using a completely different method, John [2, Theorem A] obtained the same conclusion for $M/m < \sqrt{2} = 1.414\ldots$. Although John's proof is valid in Hilbert

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* The results contained in this paper were obtained while the author was a guest at the University of Bonn.

spaces and ours can be modified so as to handle the infinite dimensional case as well, we restrict our discussion to E^n in order to facilitate the exposition.

In our discussion we use the following conventions and notation : $B(a,r)$ and $\overline{B}(a,r)$ denote, respectively, the open and closed ball of radius r and center a in E^n. $B(r) = B(0,r)$ and $\overline{B}(r) = \overline{B}(0,r)$, and D is the open unit disk in E^2. ∂A and $\overline{A}$ denote, respectively, the boundary and closure of the set A. If C is a curve in E^n its length will be denoted by $\lambda(C)$. All curves that we consider are rectifiable. If C is a curve in E^n parametrized by $\varphi(t)$ and if f is a mapping in E^n, then by $f(C)$ we shall mean the curve parametrized by $f(\varphi(t))$. For the sake of brevity we shall not always distinguish explicitly between a curve or surface as such and the set of points that lie on it. This should cause no confusion. Finally, we mention that all mapping considered are continuous.

Let C be a closed curve in E^n. We say that $\varphi : \overline{D} \to E^n$ represents a surface which spans C if the restriction of φ to ∂D is a parametrization of C. Let C be a simple closed curve in E^n and let p be any point on C. The set of straight line segments joining p to each of the other points on C is a cone which can be viewed as a surface spanning C. This fact will be used later on. Let $A \subset E^n$ be any point set and let C again be a closed curve in E^n. We say that C links with A if $A \cap C = \phi$ and if each surface which spans C intersects A. The following lemma is probably quite trivial within the context of the theory of linking. Its proof is included for the sake of completeness and to emphasize its elementary nature.

LEMMA 1. Let $n \geq 3$. Let $\overline{B}(r)$ be a ball in E^n and let $f : \overline{B}(r) \to E^n$ be a mapping whose restriction to $B(r)$ is a homeomorphism. Let a and b be two distinct points on $\partial B(r)$ for which $f(a) = f(b)$. Let C_1 and C_2 be two simple curves which connect a and b on $\partial B(r)$ and in $B(r)$, respectively. Let $A \subset B(r)$

be closed. If the simple closed curve $C = C_1 \cup C_2$ *links with* A *, then* $f(C_2)$ *, links with* $f(A)$ *.*

Proof : Since A is compact, there is a number $s < r$ for which $A \subset B(s)$. We claim that there is a surface spaning $f(C_1)$ which lies outside of $B(s)$ and which therefore does not touch $f(A)$. First of all, there exists a surface T spanning $f(C_1)$ which does not contain all of $f(B(s))$. Thus there is a point w in $f(B(s))-T$. Since the restriction of f to $\overline{B}(s)$ is a homeomorphism, it is clear that there is a mapping on $E^n - \{w\}$ which coincides with the identity outside of $f(B(s))$ and maps $f(B(s)) - \{w\}$ onto $f(\partial B(s))$. Application of this mapping to the surface T gives us the desired surface.

Now assume that S is a surface that spans $f(C_2)$. Putting this surface together, in the obvious manner, with the surface shown to exist in the last paragraph, we get a surface S' which spans $f(C)$ and which touches $f(A)$ if and only if S does . Thus it suffices to prove that $f(C)$ links with $f(A)$. It is clear that the properties of linking and non-linking of a curve with a point set are preserved when the curve is continuously deformed without touching the point set. Since we can deform C to a simple closed curve lying in $B(r)$ while staying away from A, it is sufficient to prove that if C is a simple closed curve in $B(r)$ that links with A, then $f(C)$ links with $f(A)$. We do this.

Let C be a simple closed curve in $B(r)$ that links with A. There exists a number t, $s < t < r$, for which $C \subset B(t)$. Let $\varphi : \overline{D} \to E^n$ represent a surface spanning $f(C)$. Let S^n denote the unit sphere in E^{n+1} and let $p \in S^n$. Let $g: B(r) \to S^n - \{p\}$ be a homeomorphism for which $g(x) \to p$ uniformly as $x \to \partial B(r)$ and for which $g(B(t))$ is a hemisphere. If $k(x) = g(f^{-1}(x))$ for x in $f(B(r))$ and $k(x) = p$ otherwise, then $k: E^n \to S^n$ is continuous.

By approximating φ by "nice" mapping we may assume that $\varphi(\overline{D})$ contains no open set in E^n and hence that there is a point q in $S^n - (g(\overline{B}(t)) \cup k(\varphi(\overline{D})))$. It is then clear that there exists a mapping $h: S^n - \{q\} \to g(\overline{B}(t))$ which is a homeomorphism on $g(B(t))$ and which maps $S^n - \{q\} - g(B(t))$ onto $g(\partial B(t))$. If we consider $k^{-1} \circ h \circ k \circ \varphi: \overline{D} \to E^n$, we have a surface which spans $f(C)$, lies in $f(B(r))$ and touches A if and only if $\varphi(\overline{D})$ does. Upon applying f^{-1} we see that indeed $\varphi(\overline{D})$ touches A and we are done. Q.E.D.

In the proof of the next lemma and also in that of the theorem itself we shall need the following idea. Let H be any closed half plane lying in E^n. We define a mapping $\Pi_H: E^n \to H$ which might be described as a "cylindrical projection". Let L be the straight line which forms the edge of H. Let T be any half line which is perpendicular to L and whose end point lies on L. Then Π_H maps T isometrically onto the unique half line with end point x which is perpendicular to L and lies in H. This defines Π_H uniquely. It is easy to see that this mapping is Lipschitz continuous with Lipschitz constant 1 and consequently we have $\lambda(\Pi_H(C)) \leq \lambda(C)$ for all curves C in E^n.

LEMMA 2. Let B_1 and B_2 be disjoint open balls in E^n of radius r. Let L_1 and L_2 be straight lines which pass through the centers of B_1 and B_2, respectively, and which intersect in a point p which lies outside both of the balls. Let C be a closed curve which is disjoint from $B_1 \cup B_2$, which passes through p and which intersects each L_i in some point q_i which lies on the opposite side of B_i from p. Then $\lambda(C) \geq (4 + 2\pi) r$.

Proof: We shall assume that $L_1 \neq L_2$. The case $L_1 = L_2$ can be handled either by a slight modification of the following procedure or by approximation.

We may decompose C into the union of three arcs C_1, C_2 and E, where C_i

connects p to q_i and E connects q_1 to q_2. Let P be the plane which contains L_1 and L_2. Let H_1 be the half plane contained in P bounded by L_1 which does not contain q_2 and let H_2 be the analogously defined half plane bounded by L_2. Let $C'_1 = \Pi_{H_i}(C_i)$. Clearly, C'_i connects p to q_i and lies in H_i and $\lambda(C'_i) \leq \lambda(C_i)$. Let $B'_2 = \Pi^{-1}_{H_1}(B_2 \cap H_1)$. Simple geometric considerations show that $B'_2 \subset B_2$. Since C_1 is disjoint from $B_1 \cup B_2$, it is also disjoint from $B_1 \cup B'_2$. The definition of C'_1 now implies that C'_1 is disjoint from $\Pi_{H_1}(B_1 \cup B'_2) = (B_1 \cup B_2) \cap H_1$. But since C'_1 lies in H_1, it is actually disjoint from $B_1 \cup B_2$. Similarly, C'_2 is disjoint from $B_1 \cup B_2$. Similarly, C'_2 is disjoint from $B_1 \cup B_2$. Let H be the half plane in P which is bounded by the line through the centers of the two balls and which does not contain p. It is easily seen that $E' = \Pi_H(E)$ lies in H and is disjoint from $B_1 \cup B_2$ and furthermore, $\lambda(E') \leq \lambda(E)$.

Let $C' = C'_1 \cup C'_2 \cup E'$. Then $\lambda(C') \leq \lambda(C)$. If C' is not already a simple closed curve, then it can be replaced by a simple closed curve of smaller length which connects p to q_i in H_i and q_1 to q_2 in H and which is disjoint from $B_1 \cup B_2$. We see that two disjoint circles of radius r lie inside this curve. The desired conclusion now follows from the fact that if $Y \subset E^2$ is any point set lying inside a simple closed curve C, then $\lambda(C)$ is at least equal to the perimeter of the convex hull of Y and the fact that the convex hull of two disjoint circles of radius r has perimeter at least $(4 + 2\pi)r$. Q.E.D.

In our proof of the theorem we use each of the defining conditions (U) and (L) exactly once. We do not apply these conditions directly but use the following simple consequences of them instead.

(U') Let $G \subset E^n$ be open and let C be a curve lying in G. If $f: G \to E^n$ satisfies condition (U) in G, then $\lambda(f(C)) \leq M\lambda(C)$.

(L') If $f: B(a,r) \to E^n$ is a homeomorphism which satisfies (L) in $B(a,r)$, then

$f(B(a,r)) \supset B(f(a), mr)$.

(U') follows from the fact that a mapping satisfying (U) in G is locally Lipschitz continuous with Lipschitz constant M in G. This is essentially the content of the Fundamental Lemma of Nevanlinna proved in John [1] . (L') is a very simple special case of Theorem II of that same paper .

We now begin the proof of the theorem. We shall assume throughout that $n \geq 3$. This causes no difficulties since the case $n=2$ follows trivially from the case $n=3$. Also, minor modifications of the following proof yield a direct proof for the case $n=2$. Let f be an (m,M)-isometric mapping of the ball $B(r)$ in E^n into E^n. We assume that f is not one to one and show that this implies that

$M/m > \frac{4+2\pi}{4+\pi}$.

Let r_0 be the greatest lower bound of the set of all numbers s for which f is not one to one in $B(s)$. Then $r > r_0 > 0$ and there are two distinct points a and b on $\partial B(r_0)$ for which $f(a) = f(b)$. We have $r_0 > 0$ because the mapping is a local homeomorphism. The existence of the two points a and b can be justified as follows : There exist two sequences $\{a_n\}$ and $\{b_n\}$, which can be assumed to be convergent, such that $a_n \neq b_n$, $f(a_n) = f(b_n)$ and $a_n, b_n \in B(r_0 + 1/n)$. If a and b denote the limits of these two sequences, then $a \neq b$, since f is a local homeomorphism. Also, we have $a,b \in \bar{B}(r_0)$ and the definition of the number r_0 implies that, in adition, $a,b \in \partial B(r_0)$.

Let P be the plane containing the three points a, b and 0 . We introduce a rectangular coordinate system (t_1, t_2) in such a way that $a = (z_1, z_2)$ and $b = (z_1, -z_2)$, where $0 \leq z_1$ and $0 < z_2$, and also the point $(0,0)$ coincides with the origin 0 of E^n. Let H be the right half plane $t_1 \geq 0$.

Let C_1 be the subarc of the semi-circle $\partial B(r_0) \cap H$ which connects a and b. We now claim that there exist two disjoint open disks W_1 and W_2 in H of equal radius ρ and a curve C_2 joining a to b in $B(r_0) \cap H$ for which the following conditions hold : (1) W_1 and W_2 lie inside the simple closed curve $C_1 \cup C_2$, and (2) $\lambda (C_2 < (4 + \pi)\rho$.

In the case $z_1 \geq z_2$ we consider first the open disk W_1 centered at $(z_1 - z_2/2, z_2/2)$ with radius $\rho = z_2/2$ together with W_2, its mirror image with respect to the t_1-axis. We consider the curve C consisting of the following three pieces and their mirror images with respect to the t_1-axis : the straight line segment connecting a to the point $(z_1 - z_2/2, z_2)$, the shorter of the two arcs of ∂W_1 which connect this point to $(z_1 - z_2, z_2/2)$ and the straight line segment from this last point to $(z_1 - z_2, 0)$. We have $\lambda(C) = (4 + \pi)\rho$. Since W_1 and W_2 do not touch $\partial B(r_0)$, we can move them towards $\partial B(r_0)$ and then replace C by a curve C_2 of smaller lenght for which condition (1) holds. In the case $z_1 < z_2$ we take W_1 and W_2 to be the open disks inscribed, respectively, in the first and fourth quadrants of the circle $\partial B(r_0) \cap P$. If ρ is the radius of these disks, then it can easily be seen that there always exists a curve C_2 for which conditions (1) and (2) are satisfied.

Let the centers of W_1 and W_2 be the points $w_1 = (u,v)$ and $w_2 = (u,-v)$, respectively. Let S_1 be the $(n$-$2)$-dimensional sphere with radius u and center at the point $(0,v)$ which lies in the hyperplane which is perpendicular to the t_2-axis and which passes through $(0,v)$. Let S_2 be the sphere similarly defined with center at $(0,-v)$. The curve $C_1 \cup C_2$ links with S_1 and S_2. To see this, we consider the cylindrical projection defined in the paragraph that preceeds Lemma 2. If there were a surface in E^n spanning $C_1 \cup C_2$ which were disjoint from, say S_1, then by applying Π_H to this surface we would have a surface in P which spanned $C_1 \cup C_2$ but which dit not contain the point w_1. This contradicts the fact that w_1 is inside the simple closed curve $C_1 \cup C_2$.

Let K be any cone made up of all the line segments connecting a fixed point p on $f(C_2)$ to all other points on $f(C_2)$. It was pointed out above that K is the set of points lying on a surface which spans $f(C_2)$. Applying Lemma 1 with $A = S_i$, we see that $f(C_2)$ links with $f(S_i)$. Consequently, there exists $x_i \in S_i$ for which $f(x_i) \in K$. Since $B(x_1, \rho) \subset B(r_0)$, $f(B(x_i, \rho))\ \ f(C_2) = \phi$. This follows since $f(B(r_0))\ \ f(\partial B(r_0)) = \phi$. Now we apply (L') and conclude that $B(f(x_i), m\rho) \subset f(B(x_i, \rho))$. Thus $f(C_2)\ \ B(f(x_i), m\rho) = \phi$. Let L_i be the straight line containing p and $f(x_i)$. Then by the definition of x_i and K, there is a point $q_i \in L_i\ \ f(C_2)$ which lies on the opposite side of $B(f(x_i), m\rho)$ from p. We now apply Lemma 2 with $B_i = B(f(x_i), m\rho)$ and conclude that $\lambda(f(C_2)) \geq (4 + 2\pi) m\rho$. However, since $\lambda(C_2) < (4+\pi)\rho$, we have, by (U'), that $\lambda(f(C_2)) < M(4+\pi)\rho$. Putting these two bounds for $\lambda(f(C_2))$ together we obtain $M/m > \frac{4+2\pi}{4+\pi}$, which is exactly what we had to prove.

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