DIFFERENTIAL SIMPLICITY AND A CRITERION FOR NORMALITY

by

Ives LEQUAIN

0. Introduction. Let $P$ be a point on an algebraic variety $V$ over a ground field $k$. Let $R$ be the local ring of $P$ on $V$, and let $\mathcal{D}$ be the module of derivations of $R$ into itself. If $R$ is $\mathcal{D}$-simple, then $P$ is a normal point.

Let $P$ be a point on a noetherian scheme $X$. Let $R$ be the local ring of $P$ on $X$, and let $\mathcal{D}$ be the module of derivations of $R$ into itself. If $R$ is $\mathcal{D}$-simple, then $P$ need not be anymore a normal point. We give a necessary and sufficient condition for $P$ to be normal.

1. Preliminaries. Let $R$ be a commutative ring with identity. A derivation $D$ of $R$ is a map from $R$ into $R$ such that $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$ for all $a, b \in R$.

Let $\mathcal{D}$ be a set of derivations of $R$. An ideal $I$ of $R$ is a $\mathcal{D}$-ideal if $D(I) \subseteq I$ for every $D \in \mathcal{D}$; $R$ is $\mathcal{D}$-simple if it has no $\mathcal{D}$-ideal other than $(0)$ and $(1)$. If $R$ contains the rational numbers and has no prime $\mathcal{D}$-ideal other than $(0)$ and $(1)$, then $R$ is $\mathcal{D}$-simple [2, Cor. 1.5 p. 743].

If $R$ is a domain with quotient field $K$, and if $D$ is a derivation of $R$, then $D$ can be uniquely extended to a derivation of $K$ that we also call $D$ [6, lemma p. 120]; if $T$ is any domain between $R$ and $K$ such that $D(T) \subseteq T$, we say that $D$ is regular on $T$, or that $D$ can be extended to $T$. 
We note that if $D$ is regular on a ring $T$ and if $S$ is a multiplicative system of $T$, then $D$ is regular on $T_S$. We note also that if $R$ is $\mathcal{D}$-simple, if $T$ is a ring such that $R \subseteq T \subseteq K$, and if every $D \in \mathcal{D}$ can be extended to $T$, then $T$ is $\mathcal{D}$-simple [3, Prop. 1.1, p. 216]. We shall write $D^{(0)}(x)$ to denote $x$, and for $n \geq 1$, $D^{(n)}(x)$ to denote $D(D^{(n-1)}(x))$, i.e. the $n^{th}$ derivative of $x$. If $Q$ is a prime ideal of a ring $R$ we shall write $Q^{(n)}$ to denote the $n^{th}$ symbolic power of $Q$, i.e. $Q^{(n)} = Q^nR \cap R = \{ x \in R | \exists y \in R \setminus Q \text{ such that } xy \in Q^n \}$. Of course, when $Q$ is a maximal ideal, we have $Q^{(n)} = Q^n$.

II. Case of a point on an algebraic variety. Let $P$ be a point on an algebraic variety $V$ over a ground field $k$. Let $R$ be the local ring of $P$ on $V$ and let $\mathcal{D}$ be the module of derivations of $R$. We have:

**Proposition 1.** If $R$ is $\mathcal{D}$-simple, then $P$ is a normal point, i.e. $R$ is integrally-closed.

**Proof:** $R$ is of the type $k[x_1, \ldots, x_n]_M$ where $M$ is a maximal ideal of $S = k[x_1, \ldots, x_n]$. Denoting the integral closure of $S$ by $\overline{S}$, $\overline{S}$ is a finite $S$-module [6, Theorem 9, p. 267]; thus, the conductor of $S$ in $\overline{S}$ is certainly an ideal $A \neq (0)$; then, $\overline{S}_S \setminus M = \overline{R}$ the integral closure of $R = S_M$ and the conductor of $R$ in $\overline{R}$ is $AR \neq (0)$ [6, lemma p. 269]. If the characteristic of $k$ is $p \neq 0$, then $R$ is a field [2, Theorem 1.4 p. 743] and therefore integrally closed. If the characteristic of $k$ is 0, then every $D \in \mathcal{D}$ can be extended to $\overline{R}$ [5, p. 168] so that the conductor of $R$ in $\overline{R}$ is a $\mathcal{D}$-ideal of $R$; since it is $\neq (0)$, and since $R$ is $\mathcal{D}$-simple, it has to be the ideal $(1)$, so that $R = \overline{R}$ is integrally closed.

III. Case of a point on a noetherian scheme. The conjecture that the preceding proposition should be true for a point $P$ on a noetherian scheme $X$ was given a counterexample in [2, Example 2.2, p. 746] where a noetherian, local, 1-dimensional, not integrally closed domain $R$ was constructed, and a derivation $D$ of $R$.
was defined such that $R$ was $D$-simple. Here, we shall look for conditions that make a point $P$ normal when $R$ is $D$-simple.

Thus, let $X$ be a noetherian scheme, $P$ a point on $X$, $R$ the local ring of $P$ on $X$, and $\mathcal{D}$ the module of derivations of $R$. Our assumption is that $R$ is $D$-simple. If $R$ is of characteristic $p \neq 0$, $R$ is a primary ring [2, Theorem 1.4, p. 743], hence is equal to its total quotient ring and therefore integrally closed; this case will not be anymore of interest in our considerations. Thus, we can now suppose that $R$ is a $D$-simple noetherian ring of characteristic 0; it is then a domain containing the rational numbers. [2, Cor. 1.5, p. 743]. Let $K$ be its quotient field and $\overline{R}$ its integral closure in $K$; let $\mathcal{P} = \{ \text{minimal prime ideals of } R \}$, and $R' = \bigcap_{Q \in \mathcal{P}} R_Q$. We have:

**PROPOSITION 2**: $R \subseteq R' \subseteq \overline{R}$.

*Proof*: That $R \subseteq R' = \bigcap_{Q \in \mathcal{P}} R_Q$ is clear. Now, let $Q \in \mathcal{P}$; by the Cohen-Seidenberg lying over theorem [6, Theorem 3, p. 256], there exists a prime ideal $\overline{Q}$ of $\overline{R}$ such that $\overline{Q} \cap R = Q$; by [2, Theorem 3.3, p. 749], $\overline{Q}$ is unique, and is a minimal prime; furthermore, the map $\varphi: \mathcal{P} = \{ \text{minimal prime ideals of } R \} \to \overline{\mathcal{P}} = \{ \text{minimal prime ideals of } \overline{R} \}$ defined by $\varphi(Q) = \overline{Q}$ is clearly injective since $Q = \overline{Q} \cap R$, and is surjective [2, Theorem 3.3, p. 749]. Now, since $R$ is a noetherian domain, $\overline{R}$ is a Krull ring [4, (33.10) p. 118] and $\overline{R} = \bigcap_{Q \in \mathcal{P}} \overline{R_Q}$, so that we have $R' = \bigcap_{Q \in \mathcal{P}} R_Q \subseteq \bigcap_{Q \in \mathcal{P}} \overline{R_Q} = \overline{R}$.

In [2, Example 2.2, p. 746] it was shown that $R' \not\subseteq \overline{R}$ can happen.

**LEMMA 3**. If $R$ is 1-dimensional, let $Q$ be its unique non trivial prime ideal, and let $D \in \mathcal{D}$ be such that $D(Q) \not\subseteq Q$. Then the following statements are equivalent:

(i) $P$ is a normal point on $X$, i.e. $R = \overline{R}$.
Proof: Suppose \( R \) integrally closed; then \( R \) is a rank-1-discrete valuation ring. Let \( u \) be a generator of \( Q \); since \( D(Q) \nsubseteq Q \), we have \( D(u) \notin Q \); we can suppose that \( D(u) = 1 \). If \( x \in Q^n \), we certainly have \( D(0)(x), \ldots, D(n-1)(x) \in Q \); conversely, if \( x \in Q^n \), we have \( x = u^kt \) with \( k < n \) and \( t \) a unit in \( R \); then, \( D(k)(x) = k!t + ur_k \) with \( r_k \in R \); since \( R \) is \( D \)-simple of characteristic 0, \( k!t \) is a unit in \( R \) and \( D(k)(x) \notin Q \). Thus \((i) \Rightarrow (ii)\).

Now, suppose \((ii)\) true, and let \( \mathcal{R} \) be the integral closure of \( R \). By [5, p. 168] and [3, lemma 2.2 p. 216] \( \mathcal{R} \) has only one prime ideal, thus \( \mathcal{R} \) is a rank-1-discrete valuation ring, \( D \)-simple; let \( \overline{Q} \) be its maximal ideal. For \( \mathcal{R} \), the condition \((i)\) is satisfied, hence, as was checked, we have, for every \( n \geq 1 \), \( \mathcal{Q}^n = \{ x \in \mathcal{R} \mid D(i)(x) \in \overline{Q} \text{ for } i = 0, 1, \ldots, n-1 \} \). Then, we get \( \mathcal{Q}^n \cap R = \{ x \in R \mid D(i)(x) \in \overline{Q} \cap R = Q \text{ for } i = 0, 1, \ldots, n-1 \} = Q^n \) since we suppose \((ii)\) true. Hence, \( R \) is a topological subspace of \( \mathcal{R} \) (with the \( Q \)-adic and \( \overline{Q} \)-adic topology respectively), and \( R^* \subseteq \mathcal{R}^* \) where \( R^* \) and \( \mathcal{R}^* \) are the completions of \( R \) and \( \mathcal{R} \) respectively. By [1, p. 330], \( \mathcal{R}^* \) has no nilpotent element other than 0, hence \( R^* \) has no nilpotent element other than 0 either, and again by [1, p. 330], \( \mathcal{R} \) is a finite \( R \)-module, and the conductor \( C \) of \( R \) in \( \mathcal{R} \) is different from (0). But \( C \) is a \( D \)-ideal and \( R \) is \( D \)-simple; thus \( C = (1) \) and \( R = \mathcal{R} \).

**LEMA 4.** If \( Q \) is any prime ideal of \( R \), and \( D \) any element of \( \mathcal{D} \), the following statements are equivalent:

\[(i) \quad Q^nR_Q = \{ x \in R_Q \mid D(i)(x) \in QR_Q \text{ for } i = 0, 1, \ldots, n-1 \}\]

\[(ii) \quad Q^nR_Q \cap R = \{ x \in R \mid D(i)(x) \in Q \text{ for } i = 0, 1, \ldots, n-1 \}\]

**Proof:** This is an easy computation that we shall omit.

**THEOREM 5:** The following statements are equivalent:
(i) $P$ is normal on $X$, i.e. $R = \bar{R}$

(ii) $R = R'$ and $\forall Q \in \mathcal{P}, \exists D \in \mathcal{D}$ such that $\forall n \geq 1$, the $n^{th}$ symbolic power $Q^{(n)}$ of $Q$ is equal to

$$\{ x \in R \mid D^{(i)}(x) \in Q \text{ for } i=0,1,\ldots,n-1 \}.$$

**Proof:** Note that $R$ is integrally closed if and only if $R = \bigcap Q R = R'$ and $R_Q$ is integrally closed for every $Q \in \mathcal{P}$; then apply lemmas 3 and 4.

**Remark:** When $R$ is a $\mathcal{D}$-simple noetherian ring, it is not known if $R \subset R' = \bigcap R_Q$ can happen, i.e. equivalently if $R$ can have some principal ideals with some embedded associate prime. It is nor known either if $\bar{R}$ is noetherian in general.

**BIBLIOGRAPHY**


**IMPA**

Rua Luiz de Camões, 68
Rio de Janeiro, Brasil, S. A.

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