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DIFFERENTIAL SIMPLICITY AND A CRITERION FOR NORMALITY

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0. Introduction. Let P be a point on an algebraic variety V over a ground field k. Let R be the local ring of P on V, and let $\mathfrak{D}$ be the module of derivations of R into itself. If R is $\mathfrak{D}$-simple, then P is a normal point.

Let P be a point on a noetherian scheme X. Let R be the local ring of P on X, and let $\mathfrak{D}$ be the module of derivations of R into itself. If R is $\mathfrak{D}$-simple, then P need not be anymore a normal point. We give a necessary and sufficient condition for P to be normal.

I. Preliminaries. Let R be a commutative ring with identity. A derivation D of R is a map from R into R such that $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$ for all $a,b \in R$.

Let $\mathfrak{D}$ be a set of derivations of R. An ideal I of R is a $\mathfrak{D}$-ideal if $D(I) \subseteq I$ for every $D \in \mathfrak{D}$; R is $\mathfrak{D}$-simple if it has no $\mathfrak{D}$-ideal other than (0) and (1). If R contains the rational numbers and has no prime $\mathfrak{D}$-ideal other than (0) and (1), then R is $\mathfrak{D}$-simple [2, Cor. 1.5 p. 743].

If R is a domain with quotient field K, and if D is a derivation of R, then D can be uniquely extended to a derivation of K that we also call D [6, lemma p. 120]; if T is any domain between R and K such that $D(T) \subseteq T$, we say that D is regular on T, or that D can be extended to T.

We note that if D is regular on a ring T and if S is a multiplicative system of T, then D is regular on T_S. We note also that if R is $\mathfrak{D}$-simple, if T is a ring such that $R \subseteq T \subseteq K$, and if every $D \in \mathfrak{D}$ can be extended to T, then T is $\mathfrak{D}$-simple [3, Prop. 1.1, p. 216]. We shall write $D^{(0)}(x)$ to denote x, and for $n \geq 1$, $D^{(n)}(x)$ to denote $D(D^{(n-1)}(x))$, i.e. the n^{th} derivative of x. If Q is a prime ideal of a ring R we shall write $Q^{(n)}$ to denote the n^{th} symbolic power of Q, i.e. $Q^{(n)} = Q^n R_Q \cap R = \{x \in R \mid \exists\, y \in R \setminus Q$ such that $xy \in Q^n\}$. Of course, when Q is a maximal ideal, we have $Q^{(n)} = Q^n$.

II. Case of a point on an algebraic variety. Let P be a point on an algebraic variety V over a ground field k. Let R be the local ring of P on V and let $\mathfrak{D}$ be the module of derivations of R. We have :

PROPOSITION 1. If R is $\mathfrak{D}$-simple, then P is a normal point, i.e. R is integrally-closed.

Proof: R is of the type $k[x_1, \ldots, x_n]_M$ where M is a maximal ideal of $S = k[x_1, \ldots, x_n]$. Denoting the integral closure of S by $\bar{S}$, $\bar{S}$ is a finite S-module [6, Theorem 9, p. 267]; thus, the conductor of S in $\bar{S}$ is certainly an ideal $A \neq (0)$; then, $\bar{S}_{S \setminus M} = \bar{R}$ the integral closure of $R = S_M$ and the conductor of R in $\bar{R}$ is $AR \neq (0)$ [6, lemma p. 269]. If the characteristic of k is $p \neq 0$, then R is a field [2, Theorem 1.4 p. 743] and therefore integrally closed. If the characteristic of k is 0, then every $D \in \mathfrak{D}$ can be extended to $\bar{R}$ [5, p. 168] so that the conductor of R in $\bar{R}$ is a $\mathfrak{D}$-ideal of R ; since it is $\neq (0)$, and since R is $\mathfrak{D}$-simple, it has to be the ideal (1), so that $R = \bar{R}$ is integrally closed.

III. Case of a point on a noetherian scheme. The conjecture that the preceeding proposition should be true for a point P on a noetherian scheme X was given a counterexample in [2, Example 2.2, p. 746] where a noetherian, local, *1*-dimensional, not integrally closed domain R was constructed, and a derivation D of R

was defined such that R was D-simple. Here, we shall look for conditions that make a point P normal when R is $\mathfrak{D}$-simple.

Thus, let X be a noetherian scheme, P a point on X, R the local ring of P on X, and $\mathfrak{D}$ the module of derivations of R. Our assumption is that R is $\mathfrak{D}$-simple. If R is of characteristic $p \neq 0$, R is a primary ring [2, Theorem 1.4, p. 743], hence is equal to its total quotient ring and therefore integrally closed ; this case will not be anymore of interest in our considerations. Thus, we can now suppose that R is a $\mathfrak{D}$-simple noetherian ring of characteristic 0; it is then a domain containing the rational numbers. [2, Cor. 1.5, p. 743]. Let K be its quotient field and $\bar{R}$ its integral closure in K; let $\mathcal{P} = \{$minimal prime ideals of $R\}$, and $R' = \bigcap_{Q \in \mathcal{P}} R_Q$. We have :

PROPOSITION 2 : $R \subseteq R' \subseteq \bar{R}$.

Proof : That $R \subseteq R' = \bigcap_{Q \in \mathcal{P}} R_Q$ is clear. Now, let $Q \in \mathcal{P}$; by the Cohen-Seidenberg lying over theorem [6, Theorem 3, p. 256], there exists a prime ideal $\bar{Q}$ of $\bar{R}$ such that $\bar{Q} \cap R = Q$; by [2, Theorem 3.3, p. 749], $\bar{Q}$ is unique, and is a minimal prime ; furthermore, the map $\varphi : \mathcal{P} = \{$minimal prime ideals of $R\} \to \bar{\mathcal{P}} = \{$minimal prime ideals of $\bar{R}\}$ defined by $\varphi(Q) = \bar{Q}$ is clearly injective since $Q = \bar{Q} \cap R$, and is surjective [2, Theorem 3.3, p. 749]. Now, since R is a noetherian domain, $\bar{R}$ is a Krull ring [4, (33.10) p. 118] and $\bar{R} = \bigcap_{\bar{Q} \in \bar{\mathcal{P}}} \bar{R}_{\bar{Q}}$, so that we have $R' = \bigcap_{Q \in \mathcal{P}} R_Q \subseteq \bigcap_{\bar{Q} \in \bar{\mathcal{P}}} \bar{R}_{\bar{Q}} = \bar{R}$

In [2, Example 2.2, p. 746] it was shown that $R' \subsetneq \bar{R}$ can happen.

LEMMA 3. If R is 1-dimensional, let Q be its unique non trivial prime ideal, and let $D \in \mathfrak{D}$ be such that $D(Q) \subsetneq Q$. Then the following statements are equivalent :

(i) P is a normal point on X, i.e. $R = \bar{R}$.

(ii) $\forall n \geq 1,\quad Q^n = \{ x \in R \mid D^{(i)}(x) \in Q \ \ for\ \ i = 0, 1, \ldots, n-1 \}$

Proof: Suppose R integrally closed ; then R is a rank-1-discrete valuation ring. Let u be a generator of Q ; since $D(Q) \not\subseteq Q$, we have $D(u) \notin Q$; we can suppose that $D(u) = 1$. If $x \in Q^n$, we certainly have $D^{(0)}(x), \ldots, D^{(n-1)}(x) \in Q$; conversely, if $x \notin Q^n$, we have $x = u^k t$ with $k < n$ and t a unit in R ; then, $D^{(k)}(x) = k!\,t + ur_k$ with $r_k \in R$; since R is D-simple of characteristic 0, $k!t$ is a unit in R and $D^{(k)}(x) \notin Q$. Thus (i) => (ii) .

Now. suppose (ii) true, and let $\overline{R}$ be the integral closure of R. By [5, p. 168] and [3, lemma 2.2 p. 216] $\overline{R}$ has only one prime ideal, thus $\overline{R}$ is a rank-1-discrete valuation ring, D-simple ; let $\overline{Q}$ be its maximal ideal. For $\overline{R}$, the condition (i) is satisfied, hence, as was checked, we have, for every $n \geq 1$, $\overline{Q}^{\,n} = \{ x \in \overline{R} \mid D^{(i)}(x) \in \overline{Q}$ for $i = 0, 1, \ldots, n-1\}$. Then, we get $\overline{Q}^{\,n} \cap R = \{ x \in \overline{R} \mid D^{(i)}(x) \in \overline{Q} \cap R = Q$ for $i = 0, 1, \ldots, n-1 \} = Q^n$ since we suppose (ii) true. Hence, R is a topological subspace of $\overline{R}$ (with the Q-adic and $\overline{Q}$-adic topology respectively), and $R^* \subseteq \overline{R}^*$ where R^* and $\overline{R}^*$ are the completions of R and $\overline{R}$ respectively. By [1, p. 330] , $\overline{R}^*$ has no nilpotent element other than 0, hence R^* has no nilpotent element other than 0 either, and again by [1, p. 330] , $\overline{R}$ is a finite R-module, and the conductior C of R in $\overline{R}$ is different from (0). But C is a D-ideal and R is D-simple ; thus $C = (1)$ and $R = \overline{R}$.

LEMMA 4. If Q is any prime ideal of R, and D any element of $\mathfrak{D}$, the following statements are equivalent :

(i) $Q^n R_Q = \{ x \in R_Q \mid D^{(i)}(x) \in QR_Q \ \ for\ \ i = 0, 1, \ldots, n-1 \}$

(ii) $Q^n R_Q \cap R = \{ x \in R \mid D^{(i)}(x) \in Q \ \ for\ \ i = 0, 1, \ldots, n-1 \}$

Proof: This is an easy computation that we shall omit .

THEOREM 5 : The following statements are equivalent :

(i) *P is normal on X, i.e.* $R = \bar{R}$

(ii) $R = R'$ *and* $\forall Q \in \mathcal{P}$, $\exists D \in \mathfrak{D}$ *such that* $\forall n \geq 1$, *the* n^{th} *symbolic power* $Q^{(n)}$ *of* Q *is equal to*

$$\{ x \in R \mid D^{(i)}(x) \in Q \quad for\ i=0, 1, \ldots, n-1 \}.$$

Proof: Note that R is integrally closed if and only if $R = \bigcap_{Q \in \mathcal{P}} R_Q = R'$ and R_Q is integrally closed for every $Q \in \mathcal{P}$; then apply lemmas 3 and 4.

Remark: When R is a $\mathfrak{D}$-simple noetherian ring, it is not known if $R \subsetneq R' = \bigcap_{Q \in \mathcal{P}} R_Q$ can happen, i.e. equivalently if R can have some principal ideals with some embedded associate prime. It is nor known either if $\bar{R}$ is noetherian in general.

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