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LINEAR FUNCTIONALS AND LOCAL MEASURES

(A version of the Riesz Representation Theorem in the context of metric spaces)

by

Jairo ÁLVAREZ

0) Introduction : The classical version of the Riesz Representation Theorem is proved in the context of locally compact Hausdorff spaces and the local compactness plays an essential role ([1]). This means, for instance, that the theorem is not true when the underlying space is a topological vector space of infinite dimension . This paper shows that it is possible to modify the classic proof to establish a natural extension of this theorem in the context of metric spaces or, more generally, in the context of paracompact spaces (see results in sections 5, 6, 7, 8). A similar extension can be obtained via the Daniell integral ([2]), using Stone theorem instead of our lemma and following the same type of construction used in the proof of our final theorem. However, taking advantage of the particular nature of a metric space, we use a weaker condition regarding the type of continuity imposed on the "representable" functionals.

The notion of local measure (see section 4) was introduced by C. Elson in [4] and we follow the same type of construction used there in theorem 4.4 to establish our final result.

1) *Conventions 1:* Let (X, d) be a metric space and W a non-empty open subset of X. Define :

- $S(f)$: support of the function f.
- $E^{\delta} = \{ y \in X \mid d(y, E) < \delta \}$, where δ is a positive number and E a subset of X.
- $W_{\delta} = \{ y \in W \mid \exists \rho > \delta \quad B_{\rho}(y) \subset W \}$, where δ and ρ are positive numbers.
- A non empty subset E of X is said to be properly bounded in W if there exists $\delta > 0$ such that diameter of E^{δ} is finite and $E^{\delta} \subset W$.
- $C_o(W) = \{ f : X \to \mathbb{R} ,\ f$ bounded, continuous and $S(f)$ properly bounded in $W \}$.
- A sequence of functions is said to converge uniformly in a local sense in X, if every element of X has an open neighborhood in which the sequence converges uniformly. Note that if X is a separable metric space this is equivalent to say that there exists a countable open covering (V_i) of X such that in each V_i the sequence converges uniformly.
- A sequence of functions in $C_o(W)$ is said to be C-convergent if it is uniformly convergent in a local sense in X and the sequence is uniformly bounded. Note that $C_o(W)$ is not closed with respect to this convergence.
- A sequence of functions (f_n) in $C_o(W)$ is said to be C_o-convergent or simply that it is convergent in $C_o(W)$, if it is C-convergent and besides $\bigcup_{n>0} S(f_n)$ is closed under this convergence. If f is the function in $C_o(W)$ defined by the limit of a sequence (f_n) converging in $C_o(W)$, we write $f_n \to f(C_o(W))$.
- $C(W)^*$: space of linear functionals on $C_o(W)$ continuous with respect to the C-convergence.
- $C_o(W)^*$: space of linear functionals on $C_o(W)$ continuous with respect to the convergence in $C_o(W)$. Note that $C(W)^* \subset C_o(W)^*$.
- A partition of unity on W of class $C(W)$ w.r.t to the open covering of W,

$\{U_\alpha\}_{\alpha \in \mathcal{F}}$, is a sequence of continuous functions (ψ_n) such that :

i) For every n, $\psi_n : X \to [0, 1]$

ii) For every n there exists $\alpha \in \mathcal{F}$, such that $S(\psi_n)^\delta \subset U_\alpha$, for some $\delta > 0$.

iii) For every $x \in W$ there exists an open neighborhood of x, on which, except for a finite number of indices n, $\psi_n \equiv 0$.

iv) $\sum_{i=1}^{i=\infty} \psi_i(x) = 1$ for $x \in W$.

2) *PROPOSITION 1 : Let W be a non empty open subset of a metric space X. Every countable open covering of W admits a partition of unity of class $C(W)$.*

Proof : Let (V_i) be a countable open covering for W. We may assume that all V_i are non-empty. For every V_i denote by $k(i)$ the first natural number for which $(V_i)_{1/k(i)} \neq \phi$. The family $\{ (V_i)_{1/k} : i \geq 0 , k \geq k(i) \}$ is a countable open covering for W. Enumerate it as (W_j). Since W, as a metric space, is a paracompact space, there exist a sequence (ψ_j) of continuous functions defined on W with range in $[0, 1]$ satisfying the following properties : I) For every j, $S(\psi_j) \subset W_j$. II) For every $x \in W$ there exists an open neighborhood U of x such that $U \cap S(\psi_j) = \phi$ except for finitely many indices j. III) $\sum_{i>o} \psi_i(x) = 1$ for every $x \in W$. (see [3]). From I), it follows that for every j, there exists $\delta(j) > 0$ such that $S(\psi_j)^{\delta(j)} \subset V_i \subset W$ for some i. This means that the functions ψ_i can be considered as continuous functions, defined in X, satisfying conditions i) and ii) in the definition of partition of unity of class $C(W)$, while conditions iii) and iv) coincide with properties II) and III) above. This completes the proof.

3) *PROPOSITION 2 : Let V and W be non-empty open subsets of a metric space X such that $V \subset W$. If φ is a function in $C_o(W)$ supported in V and (ψ_n) is a partition of unity on V of class $C(V)$, then the sequence $(\varphi_n = \sum_{k=1}^{n} \varphi \psi_k)$ converges in $C_o(W)$ to φ. If $\varphi \geq 0$ the convergence is monotone (in-*

creasing).

Proof: For every $x \in S(\varphi)$ it is possible to find a ball $B_{r(x)}(x)$ such that except for a finite number of ψ_k, $B_{r(x)}(x) \cap S(\psi_k) = \phi$. This means that for large k, $\varphi_k = \varphi$ on $B_{r(x)}(x)$, ie, the sequence (φ_k) converges uniformly to φ on $B_{r(x)}(x)$. If $x \notin S(\varphi)$ then there exists $\delta(x)$ such that $B_{\delta(x)}(x) \cap S(\varphi) = \phi$ and hence $\varphi_k = \varphi = 0$ on $B_{\delta(x)}(x)$ for every k. In conclusion, (φ_k) converges to φ uniformly in a local sense. It is clear that : $\bigcup_{k>o} S(\varphi_k) \subset S(\varphi)$ and hence $\bigcup_{k>o} S(\varphi_k)$ is a properly bounded subset of W. Since it is immediate that the convergence is bounded it follows that $\varphi_k \to \varphi\ (C_o(W))$. If $\varphi \geq 0$ then $0 \leq \varphi_n \leq \varphi_{n+1} \leq \varphi$ and so the convergence in this case is increasingly monotonic.

4) *Conventions 2:* Let (X, d) be a metric space and W a non-empty open subset of X.

For a subset E of X, at $\mathcal{B}(E)$ denote the Borel σ-algebra w.r.t the topology induced on E.

A natural measure in W is a non-negative measure defined on $\mathcal{B}(W)$ that takes finite values on properly bounded subsets of W and is regular in the following sense :

$$\mu(E) = \inf\{\mu(V) \mid E \subset V \subset W,\ V \text{ open subset}\},\ E \in \mathcal{B}(W).$$

$$= \sup\{\mu(K) \mid K \subset E,\ K \text{ closed subset}\}$$

A local measure ν in W is a real valued set function defined on $\{\bigcup \mathcal{B}(V),\ V$ open subset properly bounded in $W\}$, such that its restriction to any of the σ-algebras $\mathcal{B}(V)$ defines a real measure denoted by ν_V. If A and B are open subsets properly bounded in W it is clear that $\nu_A = \nu_B$ on $\mathcal{B}(A \cap B)$. It is easy, therefore, to give a meaning to the expresion $\int_W f d\nu$, for any a function f in $C_o(W)$. Note that if $W = X$ and X is bounded, a local measure in X is a real measure and the previous symbol becomes the usual integral.

Using previous definition it is easy to see that a local measure in W can be regarded as an element of $C_o(W)^*$. Our main result in this paper shows that the converse is also true. We also establish that every local measure is a weak difference of two natural measures.

5) *LEMMA : (Riesz Representation for non-negative linear functionals). Let W be a non-empty open subset of a metric space X. If T is a non negative linear functional on $C_o(W)$ such that $\lim_{n\to\infty} T(f_n) = T(f)$ whenever (f_n) is an increasing sequence of non negative functions which converges in $C_o(W)$ to f, then there exists a unique natural measure ν in W such that :*

$$T(f) = \int_W f d\nu \quad , \quad f \in C_o(W)$$

If in addition $||T|| = \sup \{ T(h) \mid 0 \leq h \leq 1,\ h \in C_o(W) \} < \infty$ then ν is a finite measure. This is in particular the case when T is C-continuous.

6) *Remark :* When X is a metric space in which closed bounded subsets are compact sets the non-negativity of the linear functional T implies that $T(f_n) \nearrow T(f)$ whenever (f_n) is a sequence of non negative functions which converge monotonically increasing to f in $C_o(W)$ and hence the previous lemma applies. In fact, take $g \in C_o(W)$ such that $0 \leq f - f_n \leq g$ for every n. Because of the compactness of $S(f)$ the convergence of (f_n) to f in $C_o(W)$ will be uniform. Therefore for arbitrary $\epsilon > 0$, n can be taken large enough such that $f - f_n \leq \frac{\epsilon}{T(g)} g$. Consequently $0 \leq T(f) - T(f_n) \leq \epsilon$, i.e. $T(f_n) \nearrow T(f)$.

Proof : Our proof proceed as in the classical case. The main difference being in the method of establishing the sub-additivity for the outer measure leading to the definition of ν.

DEFINITION 1 : **If** V **is an open subset of** W. **Define :**

$$\mu(V) = \begin{cases} sup\{ T(f) : f \in C_o(V),\ 0 \leq f \leq 1 \}, & \text{if } V \neq \phi \\ 0, & \text{if } V = \phi \end{cases}$$

DEFINITION 2 : **If** E **is a subset of** W. **Define :**

$$\mu^*(E) = inf\{ \mu(V) : E \subset V ;\ V \text{ open subset of } W \}$$

Note that :

i) **If** V **is an open subset properly bounded in** W, **then** $\mu(V) < \infty$. **In this case it is possible to find** $\psi \in C_o(W)$ **such that** $\psi = 1$ **on** V **and therefore it can be seen that** $T(\psi - f) \geq 0$ **for every** f **as specified in definition 1. Consequently** $\mu(V) \leq T(\psi) < \infty$.

ii) $\mu^*(E) < \infty$ **whenever** E **is properly bounded in** W.

iii) $\mu^*(E_1) \leq \mu^*(E_2)$, **if** $E_1 \subset E_2$

$\mu^*(V) = \mu(V)$, **if** V **is open**

iv) $\mu(V) = sup\{ T(f) : f \in C_o(W),\ 0 \leq f \leq 1,\ S(f) \subset V \}$ **(use proposition 2).**

We want to prove now that μ^* **is an outer measure. It is enough to prove that** $\mu^*(\bigcup_{i>0} E_i) \leq \sum_{i=1} \mu^*(E_i)$, **when the** E_i **are arbitrary subsets of** W **and** $\mu^*(\bigcup_{i>0} E_i) > 0$ **with** $\mu^*(E_i) < \infty$ **for every i. Given** $\epsilon > 0$, **and for every i, take** V_i **open subset of** W **such that** $E_i \subset V_i$ **and** $\mu^*(E_i) + \frac{\epsilon}{2^i} > \mu(V_i)$. **Let** c **be a real number such that** $\mu^*(\bigcup_{i>0} E_i) > c \geq 0$. **Since** $\mu(V \equiv \bigcup_{i>0} E_i) > c$ **there exists** $f \in C_o(V)$ **such that** $\mu(V) \geq T(f) > c$. **Let** (ψ_n) **be a partition of unity on** V **w.r.t** (V_i). **From proposition 2 it is known that** $(f_n = \sum_{k=1}^{k=n} f\psi_k)$ **is a monotonic increasing sequence of non-negative functions which converges in** $C_o(W)$ **to** f. **By hypothesis** $\lim_{n\to\infty} T(f_n) = T(f)$ **and hence there exists** $N > 0$ **such that for every** $n \geq N$, $T(f_n) > c$.

But $\sum_{k=1}^{n} \psi_k f$ can be written as

$$f_n = \sum_{r=1}^{r=M} \sum_{i=1}^{L(j_r)} \psi_{k_i} f$$

where $\varphi_r = \sum_{i=1}^{L(j_r)} \psi_{k_i} f \in C_o(V_{j_r})$ for some j_r. It can be seen also that $0 \leq \varphi_r \leq 1$ and consequently $T(\varphi_r) \leq \mu(V_{j_r})$. In conclusion

$$c < T(f_n) = \sum_{r=1}^{r=M} T(\varphi_r) \leq \sum_{r=1}^{r=M} \mu(V_{j_r}) \leq \sum_{r=1}^{M} \mu^*(E_{j_r}) + \frac{\epsilon}{2^{j_r}} \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon .$$

Since the analysis is valid for any $c < \mu^*(\bigcup_{i=1}^{\infty} E_i)$ and $\epsilon > 0$ is arbitrary it follows that

$$\mu^*(\bigcup_{i>0} E_i) \leq \sum_{i>0} \mu^*(E_i)$$

It is known from measure theory that the collection of μ^*-measurable sets from a σ-algebra on which μ^* is a measure. By proving that all open subsets of W are μ^*-measurable it can be concluded that all Borel subsets of W are in this σ-algebra.

We prove first that open subsets of W are inner regular. Let $\bar{\mu}(V) = sup\{\mu^*(K): K \subset V, K$ closed subset of $W\}$. It is immediate that $\bar{\mu}(V) \leq \mu(V)$. To prove that $\bar{\mu}(V) \geq \mu(V)$ assume that $\mu(V) > 0$. Let c be a real number satisfying $\mu(V) > c \geq 0$. There exists $f \in C_o(V)$, $0 \leq f \leq 1$, such that $\mu(V) \geq T(f) > c$. Let K be its support in V. If U is any open subset which contains K then f will be supported in U and from observation iv in pag. 6 , it follows that $T(f) \leq \mu(U)$. This means that $\mu(U) > c$. Therefore $\mu^*(K) \geq c$. Since c is any number less than $\mu(V)$ it follows that $\bar{\mu}(V) \geq \mu(V)$.

It is not difficult to prove now that $\mu^*(\bigcup_{i=1}^{i=N} A_i) = \sum_{i=1}^{i=N} \mu^*(A_i)$, where the A_i are disjoint open sets (or closed) and this result can be used to prove the μ^*-measurability of open sets, ie that every open set V satisfy the equality $\mu^*(A) = \mu^*(A \cap V)+$

$\mu^*(A \cap V^c)$ for any subset A of W. We omit the proof.

We will denote by ν the measure induced by μ^* on the σ-algebra of μ^*-measurable sets.

The measure ν is finite on properly bounded measurable subsets of W. This implies in particular that ν is σ-finite. Consequently, to prove regularity it is enough to prove it for properly bounded measurable subsets of W. Let E be such a set. Take closed subsets K and K_o properly bounded in W such that $E \subset K \subset K_o$, with K properly bounded in K_o. For a given $\epsilon > 0$, it is possible to find on open set V such that $K-E \subset V \subset K_o$ and $\nu(V) < \nu(K-E) + \epsilon$. The set $(K_o - V) \cap K$ is closed and it is contained in E. Moreover, $E-(K_o - V) \cap K \subset E \cap V \subset V-(K-E)$, which implies that $\nu(E) - \mu((K_o - K) \cap K) \leq \nu(V-(K-E)) < \epsilon$. Since ϵ is arbitrary it follows that $\nu(E) = sup\{\nu(K) \mid K \subset E,$ closed subset if $W\}$, which proves the inner regularity of ν. The outer regularity is immediate from its construction. In conclusion, ν is natural measure in W.

To prove the representation formula we adapt the procedure used in step X in the proof of theorem 2.14 in [1]. It is enough to prove the inequality $T(f) \leq \int_W f d\nu$, since using the linearity of T it can be concluded that $-T(f) = T(-f) \leq \int_W (-f) d\nu = -\int_W f d\nu$ and consequently $T(f) \geq \int_W f d\nu$.

Let f be a function in $C_o(W)$ with range contained in $[a, b]$. Since f can be written as $f = f \vee 0 - (-f) \vee 0$, with $f \vee 0$ and $(-f) \vee 0$ in $C_o(W)$, it can be assumed that $f \geq 0$. Let K be its support. For a given $\epsilon > 0$ choose y_i, $i = 1, 2, \ldots, n$, such that $y_i - y_{i-1} < \epsilon$ and $y_o < a < y_i < \ldots < y_n = b$. Define the sets E_i as follows

$$E_i = \{x \in W : y_{i-1} < f(x) \leq y_i\} \cap K, \quad i = 1, 2, \ldots, n.$$

Since f is continuous, f is measurable and consequently, the E_i are disjoint

Borel sets whose union is K. For every E_i it is possible to choose an open subset V_i properly bounded in W such that $V_i \supset E_i$, $\nu(V_i) < \nu(E_i) + \frac{\epsilon}{n}$ and $f(x) < y_i + \epsilon$, for every $x \in V_i$. Let $\{h_i\}_{i>0}$ be a partition of unity w.r.t $\{V_i\}_{i>0}$. From proposition 2 it follows that $\sum_{i=1}^{i=K} h_i f \in f(C_o(W))$ and for N, large enough, it holds that

$$T(f) < T(\sum_{i=1}^{i=N} h_i f) + \epsilon .$$

But $\sum_{i=1}^{i=N} h_i f$ can be written as $\sum_{k=1}^{n} \sum_{j=1}^{L_k} h_{i_j} f$, where for every k, $\sum_{j=1}^{j=L_k} h_{i_j}$ is a non negative function in $C_o(V_k)$ bounded by 1. It can be written that

$$T(f) < T(\sum_{i=1}^{i=N} h_i f) + \epsilon = \sum_{k=1}^{n} T(\sum_{j=1}^{k} h_{i_j} f) + \epsilon \leq \sum_{k=1}^{n} (y_k + \epsilon)\, T(\sum_{j=1}^{L_k} h_{i_j}) + \epsilon$$

$$\leq \sum_{k=1}^{k=n} (y_k+\epsilon)\nu(V_k) + \epsilon \sum_{k=1}^{n} (y_k+\epsilon)\nu(E_k) + \sum_{k=1}^{n} (y_k+\epsilon)\frac{\epsilon}{n} + \epsilon$$

$$\leq \sum (y_k - \epsilon)\nu(E_k) + 2\epsilon\nu(K) + (b+\epsilon) + \epsilon$$

$$\leq \sum_{k=1}^{n} \int_{E_k} f\, d\nu + \epsilon\,[\,2\nu(K) + b + \epsilon + 1\,]$$

$$\leq \int_W f\, d\nu + \epsilon\,[\,2\nu(K) + b + \epsilon + 1\,] .$$

And since ϵ is arbitrary, $T(f) \leq \int_W f\, d\nu$.

The uniqueness of ν can be proved as in theorem 2.14 in [1] taking closed sets instead of compact sets.

By definition $\nu(W) = \|T\| = \sup\{T(f) : 0 \leq f \leq 1 , f \in C_o(W)\}$, therefore ν will be a finite measure if $\|T\| < \infty$. If $T \in C(W)^*$ then $T \in C_o(W)^*$ and it will be a continuous, and hence bounded, linear functional on the vector space $C_o(W)$ normed with the sup norm. Therefore, the previous statement is valid in this case.

7) THEOREM : (Riesz Representation Theorem for C_o-continuous functionals). Let W be a non empty open subset of a metric space and T a linear functional on $C_o(W)$. If $T \varepsilon C_o(W)^*$ there exists a unique local measure μ in W, which describes T as a linear functional. Moreover, there exist natural measure μ^+ and μ in W (with $\mu \equiv$ when $T \geq 0$), such that

$$T(f) = \int_W f d\mu = \int_W f d\mu^+ - \int_W f d\mu^-, \quad f \varepsilon C_o(W)$$

If $T \varepsilon C(W)^*$ then μ is a real measure.

Proof: Let $C_o^+ = \{ f \varepsilon C_o(W) : f \geq 0 \}$. For $f \varepsilon C_o^+$ define $T^+(f) = sup\{ T(h) \mid f \geq h, h \varepsilon C_o^+ \}$. Since $T(0) = 0$, $T^+(f)$ is non-negative. If V is a properly bounded open subset of W, which contains the support of f, then T is a continuous linear functional on the vector space $C_o(V)$ normed with the sup norm. Consequently for $0 \leq h \leq f$, $| T(h) | \leq || T ||_V \, || f ||_\infty$, where $|| T ||_V = sup \{ | T(g) | : g \varepsilon C_o(V), || g ||_\infty \leq 1 \}$ is finite. Hence $T^+(f) \leq || T ||_V || f ||_\infty$. Note that if T is C-continuous on $C_o(W)$ then T is a continuous operator on $C_o(W)$ normed with the sup norm and so $T^+(f)$ can be bounded independentely of its support by $||T||_W || f ||_\infty$.

This not difficult to see that $T^+(cf) = cT^+(f)$ for $c \geq 0$ and that $T^+(f+g) \leq T^+(f) + T^+(g)$. Let us prove that T^+ is additive on C_o^+ by proving the reverse inequality. Consider $h \varepsilon C_o^+$ such that $h \leq f + g$. Define the functions r and s as follows :

$$r(x) = \begin{cases} \dfrac{f(x)\, h(x)}{f(x) + g(x)} & \text{, on } V = \{ y \varepsilon X \mid f(y) + g(y) > 0 \} \\ 0 & \text{, on } X - V \end{cases}$$

$$s(x) = \begin{cases} \dfrac{g(x)\ h(x)}{f(x)+g(x)}, & \text{on } V \\ 0, & \text{on } X-V \end{cases}$$

It can be seen that $0 \leq s \leq f$, $0 \leq r \leq g$, $s + r = h$ and that they are in C_o^+. Hence, it can be deduced that

$$T(h) = T(s) + T(r) \leq T^+(f) + T^+(g).$$

Since the analysis is valid for any $h \in C_o^+$ satisfying $h \leq f + g$ it follows that $T^+(f+g) \leq T^+(f) + T^+(g)$.

Observing that any $f \in C_o(W)$ can be expressed as $f = f^+ - f^-$ where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are functions in C_o^+, T^+ can be extended to all of $C_o(W)$ as follows

$$T^+(f) = T^+(f^+) - T^+(f^-), \text{ for } f \in C_o(W)$$

T^+ is a linear functional on $C_o(W)$. We prove its additivity. To see this observe that $f+g = (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$. Consequently $(f+g)^+ + f^- + g^+ = (f+g)^- + f^+ + g^-$. Apply T^+ to both sides an use its additivity on C_o^+ to get that $T^+(f+g) = T^+(f) + T^+(g)$.

Consider now an increasing sequence of non-negative functions (f_n) converging to f in $C_o(W)$. It is clear that $\lim_{n\to\infty} T^+(f_n)$, say a, exists and that $a \leq T^+(f)$. If $a < T^+(f)$ there exists $h \in C_o^+$, $h \leq f$ such that $a < T(h) \leq T^+(f)$. The sequence $(h \wedge f_n)$ converges to h in $C_o(W)$ and hence $\lim_{n\to\infty} T(h \wedge f_n) = T(h)$. But for every n, $T(h \wedge f_n) \leq T^+(f_n) \leq a < T(h)$ which leads to the contradictions $\lim_{n\to\infty} T(h \wedge f_n) = T(h) \leq a < T(h)$. In conclusion $\lim_{n\to\infty} T^+(f_n) = T^+(f)$, ie T^+ is a non negative linear functional satisfying all conditions required by previous lemma.

Hence, there exists a unique natural measure μ^+, associated with T^+, satisfying all properties described in the lemma. Moreover, μ^+ is a finite measure when T is C-continuous, since in this case : $\mu^+(W) \leq || T ||_W$.

We define now T^- on $C_o(W)$ as follows :

$$T^-(f) = T^+(f) - T(f) \ , \ f \in C_o(W)$$

It is not difficult to see that T^- is a non-negative linear functional satisfying all conditions required by the lemma. We write μ^- to denote the corresponding natural measure associated with it. This measure is finite when T is C-continuous since in this case $\mu^-(W) \leq 2 \ || \ T \ ||_W$.

From the definition of T^+ and T^- it follows that

$$T(f) = T^+(f) - T^-(f) = \int_W f d\mu^+ - \int_W f d\mu^- \ , \ f \in C_o(W) .$$

If $T \geq 0$ then $T^+ = T$ and we obtain $T^- \equiv 0$. Consequently $\mu^- \equiv 0$.

Let $E \in \mathcal{B}(V)$, where V is properly bounded open subset of W. Define

$$\mu(E) = \mu^+(E) - \mu^-(E) .$$

It is immediate that μ is a local measure and that the representation formula given in the statement of the theorem is valid. Let ν be another possible local measure which describes T as a linear functional on $C_o(W)$. If U is a properly bounded open subset of W, $\nu(U) = \mu(U)$ because the indicator function of U can be approximated pointwise and boundedly by functions f_n in $C_o(W)$ and with the union of their supports as a properly bounded subset of W. Hence

$$\nu(U) = \lim_{n \to \infty} \int_W f_n d\nu = \lim_{n \to \infty} \int_W f_n d\mu = \mu(U) .$$

Now, for a given properly bounded subset V of W the class of Borel subsets of V on which $\nu=\mu$ is a σ-algebra which contain all open subsets. Hence $\nu=\mu$ on $\mathcal{B}(V)$, which implies the uniqueness of μ. Since μ^+ and μ^- are finite measures when T is C-continuous it is immediate that μ is a real measure when that is the case.

8) *Remark :* As it can be seen, proposition 1 plays a central role in the proof of the results presented here. A similar proposition is valid for paracompacts spaces (see [3]) , namely that given any open contable covering (A_i) of a paracompact space X, there exists a continuous partition of unity (f_i). Therefore, if we consider the space of functions $\{ f: X \to \mathbb{R} : f$ bounded and continuous $\}$ instead of $C_o(X)$, and C-convergence instead of C_o-convergence the same proofs can be used to obtain the corresponding results expressed in the second part of both the lemma and the theorem.

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Departamento de Matemáticas
Universidad del Valle
Cali, Colombia, S. A.

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