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POLYNOMIAL INVARIANTS OF BOUNDARY LINKS

by

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Dedicated to Professor H. Yerly

§ 0. *Introduction.* An m-link is a (smooth, polygonal) embedding $l: mS^1 \to S^3$ of the disjoint[(1)] union of m circles $S_1^1+\cdots+S_m^1$ into S^3.

If l extends to an embedding $V_1+\cdots+V_m \to S^3$ of the disjoint union of m-surfaces V_i^2 with $\partial V_i = S_i^1$, l is called a *boundary* m-link, and $\{V_i\}$ is called a collection of *Seifert surfaces* for l.

The purpose of this note is to give an algebraic description of some properties of boundary m-links.

Let X be the space obtained from S^3 by removing the interior of m solid tori $T_1,\dots,T_m$ with cores $l(S_1^1),\dots,l(S_m^1)$ respectively; X is a compact 3-manifold with boundary $\partial T_1+\cdots+\partial T_m$, and of the homotopy type of the complement $S^3 - Im(l)$. By abuse of language we call X the *complement* of l. The fundamental group π of X is the *group* of l.

Let x_i (resp. y_i) be the meridian (resp. longitude) of ∂T_i. The image (*via* $\partial T_i \subset X$) μ_i (resp. λ_i) of x_i (resp. y_i) in π is called the i[th] meridian

(1) The plus sign + stands for disjoint union.

(resp. longitude) of l. The pairs (μ_i, λ_i) are determined up to simultaneous conjugation.

Finally, let $f: F \to \pi$ be the map from F, the free group on $\mu_1, \ldots, \mu_m$, into π defined by $(x_1 \vee \ldots \vee x_m) \subset X$. In [2] we prove(2)

PROPOSITION 1. An m-link is boundary if and only if f is a retraction, i.e., if there exists an exact sequence

$$(0) \qquad 1 \to K \to \pi \overset{p}{\to} F \to 1$$

and $pf = 1_F$.

The group K is then π_ω, the intersection of the members π_i of the lower central series of π [8].

The author is very grateful to Henri Yerly from whom he learned mathematics. This debt has always been uppermost in his mind. With this contribution to the *Revista* he wishes to express thanks to his teacher.

§ 1. *The Fundamental Group*. If l is a boundary link, let $\{V_i\}$ a collection of Seifert surfaces for it. Notice $\{V_i\}$ is *not* unique.

Define Y, of the homotopy type of $S^3 - \cup V_i$, to be the space obtained from S^3 by *cutting along* the V_i [6]. Then Y is a compact manifold with boundary $\sum_{i=1}^{m} (V_{i0} \cup V_{i1})$, where $V_{it} = V_i$ $(i=1, \ldots, m;\ t=0, 1)$ and $V_{i0} \cap V_{i1} = l(S_i^1)$.

For each $g \in Z^m$, let $Y(g)$ be a copy of $Y - Im(l)$ which is an open manifold with boundary

$$\Sigma_i (Int\ V_{i0}(g) + Int\ V_{i1}(g)).$$

Let, on the other hand, $\tilde{X}$ be the covering space of X associated to the

(2) This result is partially stated by Smythe in Boundary links, Wisconsin Topology Seminar, Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N.J. 1965.

commutator subgroup π_2 of π. Since the sequence

(1) $$1 \to \pi_2 \to \pi \to \mathbb{Z}^m \to 0 ,$$

is exact, it follows [2], that $\tilde{X}$ is obtained from $\cup_{g \in \mathbb{Z}^m} Y(g)$ by identifying $Int\ V_{i0}(g+\epsilon_i)$ to $Int\ V_{i1}(g)$, where $\{\epsilon_i\}$ is the canonical basis for $\mathbb{Z}^m$. We want to find a presentation for $\pi_2 = \pi_1(\tilde{X})$; in order to achieve this we use Neuwirth's technique as described in [10; th. 4.5.1].

Let $F(S)$ be a free group with generators S. A Schreier system T is a nonempty subset F such that if $g \neq 1$ belongs to it, so does g', where g' is defined by writing $g = s_1^{n_1} \dots s_k^{n_k}$, $s_i \in S$ and

$$g' = \begin{cases} gs_k & \text{if} \quad n_k < 0 \\ gs_k^{-1} & \text{if} \quad n_k > 0 \end{cases}$$

Let W be the wedge of a collection of circles indexed by S, then $\pi_1(W,x)=F(S)$. If $G \subset F(S)$ is a subgroup, there exists a cover $\tilde{W}$ of W such that $\pi_1(\tilde{W},\tilde{x}) = G$. There is a one-to-one relation between the maximal trees in W and the Schreier systems T of $F(S)$ which contain exactly one element from each coset $G\alpha$ of G. If $\varphi(\alpha)$ is such an element then

LEMMA 1. G is the free group on the generators

$$gs(\varphi(gs))^{-1}, g \in T, \ s \in S .$$

For a proof see [8]; T is called a Schreier system for $F(S)/G$. Let now $H_i = \pi_1(V_i)$ and $G^{(g)} = \pi_1(Y(g))$ the latter having a presentation

$$< Y_1^{(g)}, \dots, Y_\alpha^{(g)} : R_1^{(g)}, \dots, R_\beta^{(g)} > ;$$

we have maps $\nu_{it}^{(g)} : H_i \to G^{(g)}$ given by the inclusions $V_{it}(g) \subset Y(g)$. A Schreier system for $F(M)$ over its commutator subgroup is the set of elements $g = \alpha_1^{g_1} \dots \alpha_m^{g_m}$ which we identify to $\mathbb{Z}^m$ by $g \to \Sigma g_i \epsilon_i$. By the lemma the commutator of $F(m)$ is the free group in the set E of elements of the form

$$(2) \qquad \alpha = \alpha_1^{g_1} \dots \alpha_m^{g_m} \alpha_i (\alpha_1^{g_1} \dots \alpha_i^{g_i+1} \dots \alpha_m^{g_m})^{-1}, \quad 1 \leq i \leq m.$$

Take the space W obtained by identifying $Int\, V_{i1}(\Sigma_{j=1}^{i} g_j \epsilon_j)$ to $Int\, V_{i0}(\Sigma_{j=1}^{i} g_j \epsilon_j + \epsilon_i)$ for all i and $g_j \epsilon \mathbb{Z}$. Observe that each $Y(g) \subset W$ and that $Y(g) \cap Y(h)$ has ≤ 1 connected component in W. This can be seen in a very simple way :

The wedge $C_m = (x_1 \vee \dots \vee x_m)$ is contained in X; let $\tilde{C}_m$ be the universal abelian cover of C_m; $\tilde{C}_m \subset \tilde{X}$ and W is the space constructed by placing the $Y(g)$ at the vertices of the maximal tree T' of $\tilde{C}_m$ associated to the above Schreier system and identifying the $V_{it}(g)$ as prescribed by the edges of T'.

The group $\pi_1(W)$ is then a (weak) tree product,

$$< Y_j^{(g)} : R_k^{(g)},\ \nu_{i1}^{(\omega_i)} = \nu_{i0}^{(\omega_i + \epsilon_i)} >.$$

Here $1 \leq i \leq m, g \epsilon \mathbb{Z}^m$ and ω_i is any element of the form $\Sigma_{j=1}^{i} g_j \epsilon_j$. This can be seen by thinking that the groups $G^{(g)}$ are in the vertices of the universal abelian cover of K and that the amalgamations correspond to the edges of the maximal tree. The remaining amalgamations in W, necessary to obtain X, correspond to the elements of E : if $\alpha \epsilon E$ is written as in (2), the corresponding identification is that of $Int\, V_{i1}(\Omega_i(\alpha))$ to $Int\, V_{i0}(\Omega_i(\alpha) + \epsilon_i)$ where $\Omega_i(\alpha) = \Sigma_{j=1}^{m} g_j \epsilon_j$; thus

PROPOSITION 2. The subgroup $\pi_1(\tilde{X})$ can be expressed as

(3) $< E, Y_j^{(g)} : R_k^{(g)}, \alpha \nu_{i0}^{(\Omega_i(\alpha)+\epsilon_i)} \alpha^{-1} = \nu_{i1}^{(\Omega_i(\alpha))}, \alpha \in E >$

Proof. In fact by the Van Kampen theorem, for every new identification in W we introduce one generator $a \in E$ and one relation, namely

$$\alpha \nu_{i0}^{(\Omega_i(\alpha)+\epsilon_i)} \alpha^{-1} = \nu_{i1}^{(\Omega_i(\alpha))} .$$

§ 2. *Polynomial Invariants* .

Let $l: mS^1 \to S^3$ be an m-link with complement X. By virtue of (1), $\mathbb{Z}^m$ acts on $\tilde{X}$ and so $H_1(\tilde{X})$ is a finitely generated module over the ring $\Lambda_m = \mathbb{Z}[\mathbb{Z}^m]$. Observe that Λ_m is isomorphic to the polynomial ring $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}]$. We distinguish two homomorphisms :

i) $\epsilon : \Lambda_m \to \mathbb{Z}$ defined by $\epsilon(t_i) = 1$, and

ii) $f \mapsto \bar{f}$ an involution on Λ_m defined by $\bar{t}_i = t_i^{-1}$.

Let now

(4) $$F_2 \xrightarrow{d} F_1 \to H_1(\tilde{X}) \to 0 .$$

be a free presentation of $H_1(\tilde{X})$, where F_1 (resp. F_2) is a free Λ_m - module of rank r (resp. g). The map d is determined (upon a choice of bases for the F_i) by a matrix M with polynomial entries. For any $k \geq 0$, consider $\Delta_k \subset \Lambda_m$, the ideal generated by the $(r-k) \times (r-k)$ *minors* of M. This ideal is independent of (4). Since Λ_m is a UFD, we can define $\lambda_k \in \Lambda_m$ to be the generator (determined up to units) of the smallest principal ideal P satisfying

$$\Delta_k \subset P \subset \Lambda_m .$$

For justifications of the above assertions, see [5]; the λ_k can be chosen

chosen so that $\epsilon\lambda_k \geq 0$. The polynomials λ_k are called the (Alexander) polynomials of l. With this terminology we can rewrite proposition 2 as follows : let l be a boundary link with Seifert surfaces V_i, suppose $Z^i_k (k=1,\ldots,2\alpha_i)$ is a base for $H_1(V_i)$. Then $H_1(Y)$ is free of rank 2α, where $\alpha = \sum_i \alpha_i$. Let $\{w_j\}$ be the base of $H_1(Y)$ *dual* to $\{Z^i_k\}$ and write

$$\nu_{it}(Z^i_k) = \Sigma_j \xi^{it}_{kj} w_j .$$

The matrix $M = \| t_i \xi^{i0}_{kj} - \xi^{i1}_{kj} \|_{i,j,k}$ can be divided into blocks M_{ij}, where M_{ii} is the $(2\alpha_i \times 2\alpha_i)$- matrix $\| t_i \xi^{i0}_{kj} - \xi^{i1}_{kj} \|_{k,j}$, which depends exclusively on the knot $l(S^1_i)$ (cf. [1, p. 153]) and, for $i \neq j$, M_{ij} is the $(2\alpha_i \times 2\alpha_j)$-matrix $(\bar{t}_i - 1) \| \nu(Z^i_k, Z^j_h) \|_{k,h}$ where $\nu(Z^i_k, Z^j_h)$ is the linking number (in S^3) of Z^i_k and Z^j_h, $(k=1,\ldots,2\alpha_i,\ h=1,\ldots,2\alpha_j)$. Let T be the cokernel of M, we have

COROLLARY 3. If l is a boundary m-link (with Seifert surfaces V_i),

$$H_1(\tilde{X}) \cong \Lambda^{m-1}_m \oplus T .$$

We now prove

THEOREM 4. Let l be a boundary m-link, then $\lambda_i = 0$ for $i=1,\ldots, m-1$ and Δ_m is the principal ideal (λ_m), where

$$(i)\quad \epsilon\lambda_m = 1 \qquad\qquad (ii)\quad \bar{\lambda}_m = \lambda_m .$$

Conversely, given $\lambda \epsilon \Lambda_m$ satisfying (i), (ii), there exists a boundary m-link with $\lambda_m = \lambda$.

Proof. The first part of the proof follows from corollary 3; for the converse, we apply the method of [5] to the trivial link : write $\lambda = \Sigma\, G_g \cdot g$, where $g \in \mathbb{Z}^m$, $C_g = C_{(g^{-1})}$ (3). Only finitely many integers C_g are nonzero. Of those g for which $C_g \neq 0$ and its inverse g^{-1}, choose exactly one to obtain a set Φ. Notice $1 \in \Phi$. Let $l : mS^1 \to S^3$ be the trivial link with complement X_o. Let D be a 3-cell in X_o and in D choose a disjoint family of circles $S_g, g \in \Phi$, with linking numbers

$$\nu\,(S_g, S_1) = \nu\,(S_1, S_g) = C_g$$

and zero otherwise. Let a'_g be an arc in D connecting S_g to S_1 so that the a'_g are mutually disjoint and meet $\cup S_g$ only at its endpoints. Choose $u_g, g \in \Phi, g \neq 1$, a disjoint family of closed curves in $X_o - D$, representing the element g of $H_1(X_o) = \mathbb{Z}^m$, and bounding a disk d_g in $S^3 - D$, where the d_g are mutually disjoint. Choose now an arc from an interior point of a'_g to a point of u_g whose interior is disjoint from all a'_g, d_g. Let a_g be the connected sum of a'_g and u_g along this arc. Finally, let S be the connected sum of the S_g along the arcs a_g. We can assume that S is unknotted.

Suppose too that $l(S^1_i)$ bounds a disk D_i; S pierces D_i in pairs of points with opposite intersection numbers; this is because in taking the connected sum of the S_g along the a_g, we use ribbons with core u_g : its boundary pierces the D_i in pairs of points with the desired intersection numbers.

As in [3; p. 61] we can eliminate a *pair* of intersection points by adding a tube joining the boundaries of small disks (centered at the intersection points) removed from the D_i. At the end of the process the link l has a collection

(3) Observe that $\mathbb{Z}^m$ is written multiplicatively.

$\{V_i\}$ of Seifert surfaces (with genus depending on the number of times S intersects D_i) and $S \cap V_i = \phi$ for all i.

The next step consists on doing surgery on S^3 using a thin tubular neighborhood τ of S so that $\tau \cap V_i = \phi$. The calculation in [5; p. 81] indicates that we obtain a new link $l' : mS^1 \to \chi(S^3, \tau) = S^3$ which is a boundary link with Seifert surfaces V_i. If $\tilde{X}$ is the universal abelian cover of l',

$$H_1(\tilde{X}) = \Lambda_m^{m-1} \oplus (\Lambda_m / \lambda \Lambda_m).$$

This finishes the proof of the theorem.

§ 3. *Cobordism*.

In [4] we prove the following

LEMMA 5. Let π be a group and let $a_1, \ldots, a_m$ be elements of π satisfying

a) $\pi/\pi_2 = \mathbf{Z}^m$, *where the cosets of $a_1, \ldots, a_m \in \pi$ generate π/π_2.*

b) $H_2(\pi) = 0$.

c) π *is the smallest normal subgroups generated by the a_i.*

Then, there exists a free group L on the letters $b_1, \ldots, b_r$, and words

$$R_j = a_{i_j} w_j b_j w_j^{-1}, \tag{5}$$

where $j = 1, \ldots, r$ and $w_j \in \pi * L$, such that if R is the consequence of $\{R_1, \ldots, R_r\}$, the group $\rho = (\pi * L)/R$ contains the free group F generated by $a_1, \ldots, a_m$ as a retraction. We use lemma 5 to prove

THEOREM 6. Let $l : mS^1 \to S^3$ be an m-link with group π. Suppose F is the free group generated by the meridians μ; then l is cobordant [1] to a *boundary link if and only if the maps* $f_\#: F/F_i \to \pi/\pi_i$ *defined for* $i=2,3,\dots$ *are isomorphisms. In other words if and only if all the Milnor invariants* [9] *are zero.*

Proof. Start with a Wirtinger presentation (cf. [9])

$$\pi = < x_{ij} \mid r_{ij},\ 1 \leq i \leq m,\ \ 1 \leq j \leq \alpha_i > ,$$

where x_{ij} is represented by a loop going once around the i^th component $l(S_i^1)$ of l and $r_{ij} = u_{ij}\, x_{ij}\, u_{ij}^{-1}\, x_{i,j+1}^{-1}$, $u_{ij} = x_{pq}^{\epsilon}$, $\epsilon = \pm 1$. Consider $S_{ij} = v_{ij} x_{i1} v_{ij}^{-1} x_{i,j+1}^{-1}$, where $v_{ij} = u_{ij} \cdot u_{i,j-1} \dots u_{i1}$. Then

(6) $$\pi = < x_{ij} \mid S_{ij} > .$$

Finally, write $x'_{ij} = x_{ij} x_{i1}^{-1}$ and $x_i = x_{i1}$. The group π can be presented by

(6') $$< x_1, \dots, x_m, x'_{ij} \mid x'_{i,j+1} = [V_{ij}, x_i],\ [V_{i\alpha_i}, x_i] = 1 > .$$

Let

(6'') $$\pi^* = < x_1, \dots, x_m, x'_{ij} \mid x'_{i,j+1} = [V_{ij}, x_i],\ j < \alpha_i >$$

Assertion. If all the Milnor invariants of l are zero the natural epimorphism $\pi^* \to \pi$ induces isomorphisms

$$\pi^*/\pi^*_i \to \pi/\pi_i \qquad (2 \leq i \leq \omega).$$

This follows from the definitions [9].

Observe now that π^* satisfies the hypothesis of lemma 5 if we take $a_i = x_i$, in fact (6'') is a presentation with defect m.

Let X be the complement of l; take the product $X \times I$ and to $X \times \{1\}$

attach r 1-handles $b_1^1, \ldots, b_r^1$; $\partial (X \times I \cup \Sigma b_j^1) = X \times \{0\} \cup \partial X \times I \cup X'$, where $\pi_1(X') = \pi * L$. Let γ_j be the loop in X' describing the word R_j of (5). Along tubular neighborhoods of the γ_j, attach 2-handles b_i^2. Notice that $\partial X = \Sigma (S_i^1 \times S^1)$ and that, if we attach $\Sigma (S_i^1 \times D^2 \times I)$ to $X \times I \cup \Sigma b_j^1$ along $\partial X \times I$, the words R_j isotop to the cores b_j of the b_j^1. As a result $X \times I \cup \Sigma b_j^1 \cup \Sigma b_j^2$ is a cobordism (modulo boundary) of X and a link complement X'' with fundamental group $\pi'' = (\pi * L)/R$.

There is a natural epimorphism $p \to \pi''$, where $p = (\pi *_* L)/R$. Further, by our assertion $p/p_i \to \pi''/\pi''_i$ is an isomorphism for $i = 1, 2, \ldots, \omega$. By lemma 5, $p/p_\omega \cong F$ and so $\pi''/\pi''_\omega \cong F$ and, by proposition 1, X'' is the complement of a boundary link. This completes the proof.

One natural question arises : which boundary links are null-cobordant ?

A necessary condition comes from observing that if $h : (mS^1) \times I \to S^3 \times I$ is a null-cobordism for l, the embedding $\Sigma V_i \to S^3$ of Seifert surfaces for l extends to an embedding $\Sigma W_i^3 \to S^3 \times I$, where $\partial W_i = V_i \cup (\partial V_i \times I) \cup D_i$, where D_i is a disk. Also $W_i \cap (S^3 \times \{0\}) = V_i$ and $W_i \cap (S^3 \times \{1\} = D_i$.

Let Θ_{ij} be the $(2\alpha_i \times 2\alpha_j)$-matrix $|| \nu (Z_r^i, \nu_{jl} Z_s^j) ||$ and Θ the block matrix $||\Theta_{ij}||$. If I_{α_i} is the identity $(Z\alpha_i \times Z\alpha_i)$ - matrix, let Δ be *diag* $(t_1 I_{\alpha_1}, \ldots, t_m I_{\alpha_m})$ a diagonal matrix. If M is the presentation matrix of $H_1(\tilde{X})$

$$M = \Delta \Theta - \Theta'$$

By the arguments in [7; § 8], let $j_i : H_1(V_i) \to H_1(W_i)$ be the inclusion map; then rank $(ker\ j_i) = \alpha_i$. Notice that (with the notation of § 2) if $x \in ker\ j_i$ and $y \in ker\ j_k$, then the linking number $\nu (x, \nu_{kl} y)$ is zero (cf. [7]).

It follows that Θ_{ij} h as the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$$

where 0 is a $(\alpha_i \times \alpha_j)$ - matrix as a result .

PROPOSITION 9. Let l be a boundary m-link. If $f_i(t_i) \epsilon \mathbf{Z}[t_i, t_i^{-1}]$ is the Alexander polynomial for the knot $l(S_i^1)$ and if l is cobordant to a split link, then

$$f = \Pi f_i \cdot g \cdot \bar{g} \;,$$

where f is the Alexander polynomial for l and $g \,\epsilon\, \Lambda_m$.

§ 4. *A Question of Genus*

Let $\Theta^t = ||\Theta^t_{ij}||$ be a block matrix $(t=1,2)$. We say that Θ^2 is obtained from Θ' be an elementary *i*-expansion (or Θ^1 is an elementary *i*-reduction of Θ^2) if Θ^2_{ii} is of the form

$$\begin{pmatrix} \Theta^1_{ii} & & 0 \\ a & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \Theta^1_{ii} & b & 0 \\ & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(where a is a row vector and b a column vector), Θ^2_{ij} $(i \neq j)$ is of the form

$$(\Theta^1_{ij}\,,\, 0,\, c) \quad \text{or} \quad (\Theta'_{ij}\,,\, c, 0)\,, \text{ respectively ,}$$

(c is a column vector) and Θ^2_{ji} is of the form

$$\begin{pmatrix} \Theta^1_{ji} \\ c' \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \Theta^1_{ji} \\ 0 \\ c' \end{pmatrix} \text{, respectively .}$$

Two block matrices $A = \| A_{ij} \|$ and $B = \| B_{ij} \|$ of the same size (i.e., A_{ij} and B_{ij} are both $(a_i \times a_j)$-matrices $i,j = 1, \ldots, r$) are *block- congruent* if there exists a matrix C of the form

$$diag\,(C_1, \ldots, C_r)$$

where C_k is a nonsingular $(a_k \times a_k)$-matrix such that

$$B = CAC'$$

Two block matrices Θ^1 and Θ^2 are *S*- equivalent if Θ^2 can be obtained from Θ^1 be a finite series of *i*-expansions, *i*-contractions and block congruences. (cf. [7]).

As in [11 ; p. 484] every *S*-equivalence class of block matrices has a representative whose diagonal blocks are non-singular . We call it a *reduced* matrix.

PROPOSITION 8 . Let l be a boundary m-link (with Seifert surfaces V_i of genus α_i). If the matrix Θ is reduced the Alexander polynomial $\lambda \epsilon \Lambda_m$ of l has degree $2\alpha_i$ in the variable t_i .

This result can be reinterpreted thusly :

COROLLARY 9. Let l be a boundary m-link. If λ is the Alexander polynomial for l , λ has even degree 2α , on each variable t_i and α_i is the genus of a surface V_i belonging to a collection of Seifert surfaces for l .

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