ON THE APPROXIMATION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS
IN HILBERT SPACES

by

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§ 1. Introduction. For a real Banach space \( E \) and an integer \( m \geq 1 \), let \( C^m(E) \) denote the algebra of the real \( C^m \)-functions on \( E \).

We endow \( C^m(E) \) with the topology of uniform convergence of the functions and their derivatives of order \( \leq m \) on the compact subsets of \( E \).

In \([2]\), L. Nachbin proved that a subalgebra \( \mathcal{F} \) of \( C^m(\mathbb{R}^n) \) is dense in \( C^m(\mathbb{R}^n) \) if and only if it fulfills the following conditions:

\( (N) \) (i) For every pair of distinct points \( a_1, a_2 \in \mathbb{R}^n \), there exists an \( f \in \mathcal{F} \) such that \( f(a_1) \neq f(a_2) \).

(ii) For every \( a \in \mathbb{R}^n \), there exists an \( f \in \mathcal{F} \) such that \( f(a) \neq 0 \).

(iii) For every \( a \in \mathbb{R}^n \) and for every \( u \in \mathbb{R}^n \) with \( u \neq 0 \), there exists an \( f \in \mathcal{F} \) such that \( (Df)(a)(u) \neq 0 \).

See also \([1]\).

It seems natural to ask for similar conditions to be satisfied by a subalgebra of \( C^m(E) \) in order that it be dense in \( C^m(E) \). In this paper we give a partial
answer to that problem, for the case that $E$ is a Hilbert space $H$. Precisely, we give sufficient conditions for a subalgebra $\mathcal{F}$ of $C^1(H)$ to be dense in $C^1(H)$, and show by a simple counterexample that, if $m \geq 2$ and $H$ has infinite dimension, these conditions are not sufficient for a subalgebra of $C^m(H)$ to be dense in $C^m(H)$. (See Remark 2 at the end of the article). In particular, the algebra of the polynomials of finite type on $H$ is dense in $C^1(H)$, but not in $C^m(H)$, for $m \geq 2$. A related problem has been studied by G. Restrepo [3].

§ 2. The main theorem. Let $H$ be a separable real Hilbert space of infinite dimension, $\{ e_n; n \in \mathbb{N} \}$ an orthonormal basis of $H (\mathbb{N} = \{ 1, 2, \ldots, \})$, and for every $n \in \mathbb{N}$, $H_n$ the span of $\{ e_1, \ldots, e_n \}$ and $P_n$ the orthogonal projection of $H$ on the subspace $H_n$.

We say that a subalgebra $\mathcal{F}$ of $C^1(H)$ fulfills conditions $\left( N_{0} \right)$ if:

(i) For every pair of distinct points $a_1, a_2 \in H$, there exists an $f \in \mathcal{F}$ such that $f(a_1) \neq f(a_2)$.

(ii) For every $a \in H$, there exists an $f \in \mathcal{F}$ such that $f(a) \neq 0$.

(iii) For every $a \in H$ and for every $u \in H$ with $u \neq 0$, there exists an $f \in \mathcal{F}$ such that $\left[ (Df)(a) \right](u) \neq 0$.

(iv) There is an $M \in \mathbb{N}$ such that for every integer $n \geq M$, if $f \in \mathcal{F}$ then $f \circ P_n \in \mathcal{F}$.

Remark. We denote, as usual, by $Df$ the derivative of $f \in C^1(H)$. $Df$ is a mapping of $H$ into the dual space $H'$ of $H$ (which is isomorphic to $H$).

Example. The algebra of the polynomials of finite type on $H$ (that is, the algebra generated by the functions of the form $x \to u_1(x)^{\alpha_1} \cdots u_n(x)^{\alpha_n}$, where $u_1, \ldots, u_n \in H'$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_+$) fulfills conditions $\left( N_{0} \right)$. 

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THEOREM. Every subalgebra \( \mathcal{S} \) of \( C^1(H) \) fulfilling conditions \( (N) \) is dense in \( C^1(H) \) with the topology of the uniform convergence of the functions and their derivatives on the compact subsets of \( H \).

§ 3. Proof of the theorem.

**Lemma 1.** Let \( (x_n) \) be a sequence in \( H \), convergent to \( x \in H \). Then
\[
\lim_{n \to \infty} P_n x_n = x.
\]

**Proof.**
\[
|| P_n x_n - x || \leq || P_n x_n - P_n x || + || P_n x - x || \leq || x_n - x || + || P_n x - x || \to 0
\]

**Proposition 1.** Let \( X \) be a metric space, \( v : H \to X \) continuous and \( v_n = v \circ P_n \). Then \( v_n \) converges to \( v \) uniformly on every compact subset \( K \) of \( H \).

**Proof.** Obviously, for every \( x \in H \), \( \lim_{n \to \infty} v_n(x) = v(x) \). Suppose that the convergence is not uniform on \( K \). Then there exist an \( \varepsilon > 0 \), an increasing sequence of integers \( n_1 < n_2 < \cdots < n_k < \cdots \) and a sequence \( (x_k) \) in \( K \), such that
\[
| v_{n_k} (x_k) - v(x_k) | \geq \varepsilon, \quad \text{for} \quad k = 1, 2, \ldots \quad (\ast).
\]

Since \( K \) is compact, \( (x_k) \) contains a convergent subsequence, which we shall denote also by \( (x_k) \). Let \( x = \lim_{k} x_k \). Then, by Lemma 1, we have:
\[
\lim_{k} (P_{n_k} x_k) = x, \quad \text{hence} \quad \lim_{k} v_{n_k} (x_k) = \lim_{k} v (P_{n_k} x_k) = v(x), \quad \text{that is},
\]
\[
\lim_{k} (v_{n_k} (x_k) - v(x_k)) = v(x) - v(x) = 0, \quad \text{which contradicts (\ast)}.
\]

**Corollary.** \( P_n \) converges to the identity operator \( I \) of \( H \), uniformly on every compact subset of \( H \).

**Proof.** In Proposition 1, set \( X = H, \ v = I \).

**Proposition 2.** If \( K \) is a compact subset of \( H \), then the closure of
\[ \bigcup_{n=1}^{\infty} P_n(K) \text{ is also a compact subset of } H. \]

**Proof.** We will show that \( A = \bigcup_{n=1}^{\infty} P_n(K) \) is a precompact set.

For this, we show that every infinite subset \( E \) of \( A \) contains a Cauchy sequence. In fact, there are two possibilities for \( E \):

a) There exists an index \( n \) such that the set \( E \cap P_n(K) \) is infinite. In this case, it is obvious that \( E \) contains a Cauchy sequence.

b) There exist a sequence of indices \( n_1 < \cdots < n_k < \cdots \) and a sequence \( (x_k) \) in \( E \) such that for every \( k \), \( x_k \in P_{n_k}(K) \).

Let us choose, for every index \( k \), \( y_k \in K \) such that \( x_k = P_{n_k}(y_k) \). Since \( K \) is compact, the sequence \( (y_k) \) contains a convergent subsequence, which we shall denote also by \( (y_k) \). If \( y = \lim_{k \to \infty} y_k \), we have by Lemma 1: \( \lim_{k \to \infty} x_k = \lim_{k \to \infty} P_{n_k} y_k = y \), i.e., the sequence \( (x_k) \) is convergent in \( H \).

**Proposition 3.** Let \( \mathcal{F} \) be a subalgebra of \( C^1(H) \) fulfilling conditions \( (N_0) \). Then the algebra \( \mathcal{F}_n = \{ f \mid H_n \ : \ f \in \mathcal{F} \} \) is dense in \( C^1(H_n) \), for \( n = 1, 2, \ldots \).

**Proof.** We have to show that \( \mathcal{F}_n \) satisfies Nachbin conditions \( (N) \): (i) and (ii) are immediate. To prove (iii), we must show that for every \( x \in H_n \) and for every \( u \in H_n \) with \( u \neq 0 \), there exists an \( f_n \in \mathcal{F}_n \) such that \( [(Df_n)(x)](u) \neq 0 \).

Since \( \mathcal{F} \) verifies \( (N_0) \) (iii), there exists \( f \in \mathcal{F} \) with \( [(Df)(x)](u) \neq 0 \). Putting \( f_n = f \mid H_n \), and recalling that \( x, u \in H_n \), we have:

\[
[(Df_n)(x)](u) = [(Df)P_n x](P_n u) = [(Df)(x)](u) \neq 0.
\]

**Remark.** \( (Df_n)(x) = [D(f \circ P_n)](x) = [(Df)(P_n x)] \circ P_n \).

**Lemma 2.** For \( f \in C^1(H) \) and \( n \in \mathbb{N} \), put \( f_n = f \circ P_n \). Then the operator
$D_n$ converges to $Df$ uniformly on every compact subset $K$ of $H$.

Proof. For $x \in K$, let $\varphi_n(x) = Df(P_n x) \in H'$. Then, we have: $Df(x) = \left[ \varphi_n(x) \right] \circ P_n$. By identifying $H$ and its dual $H'$, it follows from the self-adjointness of the orthogonal projections that

$$\left[ \varphi_n(x) \right] \circ P_n = P_n (\varphi_n(x)).$$

Since for every $x \in K$ and every $n \in \mathbb{N}$,

$$\varphi_n(x) = Df(P_n x) \in (Df) \left( \bigcup_{n=1}^{\infty} P_n(K) \right),$$

it follows from Proposition 2, that the set $\Gamma = \bigcup_{n=1}^{\infty} \{ \varphi_n(x); x \in K \}$ is precompact in $H$. Then, by the corollary to Proposition 1, $P_n$ converges to the identity operator $I$ uniformly on $\Gamma$.

Since, by Proposition 1, $\varphi_n(x)$ converges to $Df(x)$ uniformly for $x \in K$, we have proved the lemma.

Proof of the main theorem. Let $K$ be an arbitrary compact subset of $H$, and for $b \in C^1(H)$, let $\Phi(b) = \sup \{ |b(x)| + ||Db(x)||; x \in K \}$, and $b_n = b \circ P_n (n \in \mathbb{N})$. We are going to show that, given $f \in C^1(H)$ and $\varepsilon > 0$, there exists $g \in \mathcal{F}$ such that $\Phi(f - g) < \varepsilon$. In fact, by Proposition 1 and Lemma 2, we can find an index $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$, $\Phi(f - f_n) < \varepsilon/2$. Now, fix an $n \geq \max \{ n_0, M \}$ where $M$ is the integer given by condition (No) (iv). Since, by Proposition 3, $\mathcal{F}_n = \{ b_n; b \in \mathcal{F} \}$ is dense in $C^1(H_n)$, there exists $g(n) \in \mathcal{F}$ such that $\Phi(f_n - g(n)) < \varepsilon/2$. By condition (No) (iv), $g(n) \in \mathcal{F}$, so we can take $g = g(n)$.

§ 4. Some remarks.

Remark 1. Clearly, condition (iv) of (No) can be replaced by the following
(iv') There exists an increasing sequence \((n_k)\) of indices such that for every \(k, \mathcal{F}_{n_k} \subset \mathcal{F}\).

**Remark 2.** If \(m \geq 2\), conditions \((N_0)\) are not sufficient for a subalgebra of \(C^m(H)\) to be dense in \(C^m(H)\). For instance, if \(f(x) = \frac{1}{2} \| x \|^2\), we have \(D^2f = I\), but if \(p\) is a polynomial function of finite type on \(H\), then for every \(a \in H\) the operator \(D^2 p(a)\) has finite rank. Consequently, the identity operator \(I\) cannot be approximated by a sequence of operators of that form.

**Remark 3.** It is easy to show that conditions \((N_0)\ (i), (ii), (iii)\) are necessary for \(\mathcal{F}\) to be a dense subalgebra of \(C^1(H)\). It would be interesting to know if they are also sufficient, that is, if condition \((N_0)\ (iv)\) in the theorem is superfluous.

**Remark 4.** Let \(H\) be a non-separable real Hilbert space. Since compact metric spaces are separable, every compact subset of \(H\) is contained in some separable subspace of \(H\). So, by modifying appropriately condition \((N_0)\ (iv)\), one gets from the theorem sufficient conditions for a subalgebra of \(C^1(H)\) to be dense in \(C^1(H)\). In particular, the algebra of the polynomials of finite type on \(H\) is dense in \(C^1(H)\).

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**REFERENCES**


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