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ON THE APPROXIMATION OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS IN HILBERT SPACES

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§ 1. *Introduction.* For a real Banach space E and an integer $m \geq 1$, let $C^m(E)$ denote the algebra of the real C^m-functions on E.

We endow $C^m(E)$ with the topology of uniform convergence of the functions and their derivatives of order $\leq m$ on the compact subsets of E.

In [2], L. Nachbin proved that a subalgebra $\mathcal{F}$ of $C^m(\mathbb{R}^n)$ is dense in $C^m(\mathbb{R}^n)$ if and only if it fulfills the following conditions:

(N) (i) For every pair of distinct points $a_1, a_2 \in \mathbb{R}^n$, there exists an $f \in \mathcal{F}$ such that $f(a_1) \neq f(a_2)$.

(ii) For every $a \in \mathbb{R}^n$, there exists an $f \in \mathcal{F}$ such that $f(a) \neq 0$.

(iii) For every $a \in \mathbb{R}^n$ and for every $u \in \mathbb{R}^n$ with $u \neq 0$, there exists an $f \in \mathcal{F}$ such that $[(Df)(a)](u) \neq 0$.

See also [1].

It seems natural to ask for similar conditions to be satisfied by a subalgebra of $C^m(E)$ in order that it be dense in $C^m(E)$. In this paper we give a partial answer to that problem, for the case that E is a Hilbert space H. Precisely, we give sufficient conditions for a subalgebra $\mathfrak{F}$ of $C^1(H)$ to be dense in $C^1(H)$, and show by a simple counterexample that, if $m \geq 2$ and H has infinite dimension, these conditions are not sufficient for a subalgebra of $C^m(H)$ to be dense in $C^m(H)$. (See Remark 2 at the end of the article). In particular, the algebra of the polynomials of finite type on H is dense in $C^1(H)$, but not in $C^m(H)$, for $m \geq 2$. A related problem has been studied by G. Restrepo [3].

§ 2. *The main theorem.* Let H be a separable real Hilbert space of infinite dimension, $\{ e_n ; n \in \mathbb{N} \}$ an orthonormal basis of $H(\mathbb{N} = \{ 1, 2, \ldots, \})$, and for every $n \in \mathbb{N}$, H_n the span of $\{ e_1, \ldots, e_n \}$ and P_n the orthogonal projection of H on the subspace H_n.

We say that a subalgebra $\mathfrak{F}$ of $C^1(H)$ fulfills conditions (N_o) if :

(i) For every pair of distinct points $a_1, a_2 \in H$, there exists an $f \in \mathfrak{F}$ such that $f(a_1) \neq f(a_2)$.

(ii) For every $a \in H$, there exists an $f \in \mathfrak{F}$ such that $f(a) \neq 0$.

(iii) For every $a \in H$ and for every $u \in H$ with $u \neq 0$, there exists an $f \in \mathfrak{F}$ such that $[(Df)(a)](u) \neq 0$.

(iv) There is an $M \in \mathbb{N}$ such that for every integer $n \geq M$, if $f \in \mathfrak{F}$ then $f \circ P_n \in \mathfrak{F}$.

Remark. We denote, as usual, by Df the derivative of $f \in C^1(H)$. Df is a mapping of H into the dual space H' of H (which is isomorphic to H).

Example. The algebra of the polynomials of finite type on H (that is, the algebra generated by the functions of the form $x \to u_1(x)^{\alpha_1} \ldots u_n(x)^{\alpha_n}$, where $u_1, \ldots, u_n \in H'$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_+$) fulfills conditions (N_o).

THEOREM. Every subalgebra $\mathcal{F}$ *of* $C^1(H)$ *fulfilling conditions* (N_o) *is dense in* $C^1(H)$ *with the topology of the uniform convergence of the functions and their derivatives on the compact subsets of* H.

§ 3. *Proof of the theorem.*

LEMMA 1. Let (x_n) *be a sequence in* H, *convergent to* $x \in H$. *Then* $\lim_n P_n x_n = x$.

Proof. $\| P_n x_n - x \| \leq \| P_n x_n - P_n x \| + \| P_n x - x \| \leq \| x_n - x \| + \| P_n x - x \| \to 0$

PROPOSITION 1. Let X *be a metric space,* $v : H \to X$ *continuous and* $v_n = v \circ P_n$. *Then* v_n *converges to* v *uniformly on every compact subset* K *of* H.

Proof. Obviously, for every $x \in H$, $\lim_n v_n(x) = v(x)$. Suppose that the convergence is not uniform on K. Then there exist an $\varepsilon > 0$, an increasing sequence of integers $n_1 < n_2 < \ldots < n_k < \ldots$ and a sequence (x_k) in K, such that

$$| v_{n_k}(x_k) - v(x_k) | \geq \varepsilon , \quad \text{for} \quad k = 1, 2, \ldots \quad (*) .$$

Since K is compact, (x_k) contains a convergent subsequence, which we shall denote also by (x_k). Let $x = \lim_k x_k$. Then, by Lemma 1, we have : $\lim_k (P_{n_k} x_k) = x$, hence $\lim_k v_{n_k}(x_k) = \lim_k v(P_{n_k} x_k) = v(x)$, that is, $\lim_k (v_{n_k}(x_k) - v(x_k)) = v(x) - v(x) = 0$, which contradicts $(*)$.

COROLLARY. P_n *converges to the identity operator* I *of* H, *uniformly on every compact subset of* H.

Proof. In Proposition 1, set $X = H$, $v = I$.

PROPOSITION 2. If K *is a compact subset of* H, *then the closure of*

$\bigcup_{n=1}^{\infty} P_n(K)$ *is also a compact subset of* H.

Proof. We will show that $A = \bigcup_{n=1}^{\infty} P_n(K)$ is a precompact set.

For this, we show that every infinite subset E of A contains a Cauchy sequence. In fact, there are two possibilities for E:

a) There exists an index n such that the set $E \cap P_n(K)$ is infinite. In this case, it is obvious that E contains a Cauchy sequence.

b) There exist a sequence of indices $n_1 < \dots < n_k < \dots$ and a sequence (x_k) in E such that for every k, $x_k \in P_{n_k}(K)$.

Let us choose, for every index k, $y_k \in K$ such that $x_k = P_{n_k}(y_k)$. Since K is compact, the sequence (y_k) contains a convergent subsequence, which we shall denote also by (y_k). If $y = \lim_k y_k$, we have by Lemma 1: $\lim_k x_k = \lim_k P_{n_k} y_k = y$, i.e., the sequence (x_k) is convergent in H.

PROPOSITION 3. Let $\mathcal{F}$ *be a subalgebra of* $C^1(H)$ *fulfilling conditions* (N_o). *Then the algebra* $\mathcal{F}_n = \{ f \mid H_n \; ; \; f \in \mathcal{F} \}$ *is dense in* $C^1(H_n)$, *for* $n = 1, 2, \dots$

Proof. We have to show that $\mathcal{F}_n$ satisfies Nachbin conditions (N): (i) and (ii) are immediate. To prove (iii), we must show that for every $x \in H_n$ and for every $u \in H_n$ with $u \neq 0$, there exists an $f_n \in \mathcal{F}_n$ such that $[(Df_n)(x)](u) \neq 0$.

Since $\mathcal{F}$ verifies (N_o) (iii), there exists $f \in \mathcal{F}$ with $[(Df)(x)](u) \neq 0$. Putting $f_n = f \mid H_n$, and recalling that $x, u \in H_n$, we have:

$$[(Df_n)(x)](u) = [(Df) P_n x)] (P_n u) = [(Df)(x)] (u) \neq 0 .$$

Remark. $(Df_n)(x) = [D(f \circ P_n)](x) = [(Df)(P_n x)] \circ P_n$.

LEMMA 2. For $f \in C^1(H)$ *and* $n \in \mathbb{N}$, *put* $f_n = f \circ P_n$. *Then the operator*

Df_n *converges to* Df *uniformly on every compact subset* K *of* H.

Proof. For $x \in K$, let $\varphi_n(x) = Df(P_n x) \in H'$.

Then, we have : $Df_n(x) = [\varphi_n(x)] \circ P_n$. By identifying H and its dual H', it follows from the self-adjointness of the orthogonal projections that

$$[\varphi_n(x)] \circ P_n = P_n(\varphi_n(x)).$$

Since for every $x \in K$ and every $n \in \mathbb{N}$,

$$\varphi_n(x) = Df(P_n x) \in (Df)\left(\bigcup_{n=1}^{\infty} P_n(K)\right),$$

it follows from Proposition 2, that the set $\Gamma = \bigcup_{n=1}^{\infty} \{\varphi_n(x) ; x \in K\}$ is precompact in H. Then, by the corollary to Proposition 1, P_n converges to the identity operator I *uniformly on* Γ.

Since, by Proposition 1, $\varphi_n(x)$ converges to $Df(x)$ uniformly for $x \in K$, we have proved the lemma.

Proof of the main theorem. Let K be an arbitrary compact subset of H, and for $h \in C^1(H)$, let $\Phi(h) = sup\{|h(x)| + \|Dh(x)\| ; x \in K\}$, and $h_n = h \circ P_n$ $(n \in \mathbb{N})$. We are going to show that, given $f \in C^1(H)$ and $\varepsilon > 0$, there exists $g \in \mathcal{F}$ such that $\Phi(f-g) < \varepsilon$. In fact, by Proposition 1 and Lemma 2, we can find an index $n_o \in \mathbb{N}$, such that for every $n \geq n_o$, $\Phi(f-f_n) < \varepsilon/2$. Now,fix an $n \geq max\{n_o, M\}$ where M is the integer given by condition (N_o) *(iv)*. Since, by Proposition 3, $\mathcal{F}_n = \{h_n ; h \in \mathcal{F}\}$ is dense in $C^1(H_n)$, there exists $g^{(n)} \in \mathcal{F}$ such that $\Phi(f_n - g_n^{(n)}) < \varepsilon/2$. By condition (N_o) *(iv)*, $g_n^{(n)} \in \mathcal{F}$, so we can take $g = g_n^{(n)}$.

§ 4. *Some remarks.*

Remark 1. Clearly, condition *(iv)* of (N_o) can be replaced by the following one :

(iv') There exists an increasing sequence (n_k) of indices such that for every k, $\mathcal{F}_{n_k} \subset \mathcal{F}$.

Remark 2. If $m \geq 2$, conditions (N_o) are not sufficient for a subalgebra of $C^m(H)$ to be dense in $C^m(H)$. For instance, if $f(x) = \frac{1}{2} \|x\|^2$, we have $D^2 f \equiv I$, but if p is a polynomial function of finite type on H, then for every $a \in H$ the operator $D^2 p(a)$ has finite rank. Consequently, the identity operator I cannot be approximated by a sequence of operators of that form.

Remark 3. It is easy to show that conditions (N_o) *(i)*, *(ii)*, *(iii)* are necessary for $\mathcal{F}$ to be a dense subalgebra of $C^1(H)$. It would be interesting to know if they are also sufficient, that is, if condition (N_o) *(iv)* in the theorem is superfluous.

Remark 4. Let H be a non-separable real Hilbert space. Since compact metric spaces are separable, every compact subset of H is contained in some separable subspace of H. So, by modifying appropiately condition (N_o) *(iv)*, one gets from the theorem sufficient conditions for a subalgebra of $C^1(H)$ to be dense in $C^1(H)$. In particular, the algebra of the polynomials of finite type on H is dense in $C^1(H)$.

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REFERENCES

1. *Glaeser, G. Algèbres et sous-algèbres de fonctions différentiables.* An. Acad. Brasil. Ci 37, 395-406 (1965).

2. *Nachbin, L. Sur les algēbres denses de fonctions différentiables sur une variété.* C. R. Acad. Sci. Paris 228, 1549-1551 (1949).

3. *Restrepo, G. An infinite dimensional version of a theorem of Bernstein.* Proc. Amer. Math. Soc. 23, 193-198 (1969).

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