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ON THE FUNDAMENTAL UNIT AND CLASS NUMBER OF CERTAIN QUADRATIC FUNCTION FIELDS, I

by

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1. *Introduction :* **Let $K = k_o(x)(\sqrt{f(x)})$ be a function field, where k_o is a finite field of characteristic not equal to 2 and $f(x)$ is a square free polynomial of even degree and leading coefficient 1. In this situation K is said to be a real quadratic extension of $k_o(x)$. Throughout we will concentrate on real quadratic extensions of $k_o(x)$ with fundamental unit ε_o such that the norm of ε_o, denoted $N\varepsilon_o$, is a square in $k_o^* = k_o - \{0\}$. Our aim is to give explicit expressions for the fundamental unit and also lower bounds for the class number of the integral closure, $k_o(x)[\sqrt{f(x)}]$, of $k_o(x)$ in $k_o(x)(\sqrt{f(x)})$ for certain kind of $f(x)$.**

Such problems have been studied, for real quadratic number fields, by various authors e.g. Hasse [*1*], Yokoi [*2*], etc.

2. *Preliminaries :* **We start by proving various results needed for our main theorems.**

PROPOSITION 2.1 : Let $\varepsilon_o = A_o + B_o\sqrt{f(x)}$ be the fundamental unit of **$K = k_o(x)(\sqrt{f(x)})$.** *If $\varepsilon = A_1 + B_1\sqrt{f(x)}$ is any non-trivial unit of K then*

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$B_o \mid B_1$ *and* $\deg A_1 \geq \deg A_o$.

Proof. By considering ε or its conjugate ε' we can write $\varepsilon = b\,\varepsilon_o^n$ with b in k_o^* and $n \geq 1$. Then

$$A_1 + B_1\sqrt{f(x)} = b\,(A_o + B_o\sqrt{f(x)})^n = b\,(A_o^n + {}^nC_2\,A_o^{n-2}\,B_o^2\,f(x) + \cdots)$$

$$+\, b\,B_o\,({}^nC_1\,A_o^{n-1} + {}^nC_3\,A_o^{n-3}\,B_o^2\,f(x) + \cdots)\sqrt{f(x)}\,.$$

Comparing both sides we see that $B_o \mid B_1$. But then

$$\begin{aligned} \deg A_1 &= \tfrac{1}{2}\deg(B_1^2\,f(x) + N\,\varepsilon) \\ &= \tfrac{1}{2}\deg(B_o^2\,f(x) + N\,\varepsilon_o) \\ &= \deg A_o\,, \end{aligned}$$

whence the assertion.

PROPOSITION 2.2. Let ε_o *be the fundamental unit of* $K = k_o(x)(\sqrt{f(x)})$ *such that* $N\,\varepsilon_o = a^2$. *Then* K *is generated over* $k_o(x)$ *by a function of the form* $\sqrt{g(x)^2 - a^2}$. *Conversely, if* K *is generated over* $k_o(x)$ *by a function of the form* $\sqrt{g(x)^2 - a^2}$ *then* $\varepsilon_o = g(x) + \sqrt{g(x)^2 - a^2}$ *is the fundamental unit of* K *and* $N\,\varepsilon_o = a^2$.

Proof. Suppose that K is generated by a function of the form $\sqrt{g(x)^2 - a^2}$. Let $\varepsilon_o = A_o + B_o\sqrt{g(x)^2 - a^2}$ be its fundamental unit. Now, since $g(x) + \sqrt{g(x)^2 - a^2}$ is a unit of K with norm a^2, by Proposition 2.1, we see that B_o is an element of k_o. Let $B_o = b$. We assert that $N\,\varepsilon_o = a^2 b^2$ and $A = \pm\, b\, g(x)$. Suppose that $N\,\varepsilon_o = a_1$ i.e. $A_o^2 - b^2(g(x)^2 - a^2) = a_1$. Writing $A_o = h(x) + G$, we get

$$(h(x) + c)^2 = b^2 g^2(x) + a_1 - a^2 b^2 \qquad (2.1)$$

$$\text{i.e.}\quad (bg(x) - h(x))\,(bg(x) + h(x)) = 2c\,h(x) + a^2 b^2 - a_1 \qquad (2.2)$$

Comparing both sides of (2.2), we see that either $bg(x) - h(x)$ or $bg(x) + h(x)$ is in k_o^*.

Suppose that $bg(x) - h(x) = c_1$. Substituting in (2.2), we get $c = c_1$ and then (2.1) gives $a_1 = a^2 b^2$ and $c = 0$. Thus the assertion follows.

The converse is trivial.

DEFINITION 2.1. We will say that $K = k_o(x)(\sqrt{f(x)})$ is of *(*) type* if $f(x) = g(x)^2 - a^2$ for some $g(x)$ in $k(x)$ and a in k.

PROPOSITION 2.3. *Let* $\varepsilon = A + p(x) \cdot \sqrt{f(x)}$, *where* $p(x)$ *is a prime polynomial, be a unit of* $K = k_o(x)(\sqrt{f(x)})$ *such that* $N\varepsilon = a^2$ *for some* a *in* k_o^*. *Then* ε *is the fundamental unit of* K *if and only if* K *is not of (*) type.*

Proof. Suppose that K is not of (*) type. Let $\varepsilon_o = A_o + B_o\sqrt{f(x)}$ be its fundamental unit. By Proposition 2.1, $B_o \mid p(x)$. Since K is not of (*) type, $B_o = bp(x)$ for some b in k_o^* follows by Proposition 2.2. Let $N\varepsilon_o = A_o^2 - b^2 p^2(x) f(x) = a_1$. We also above $A^2 - p^2(x) f(x) = a^2$. Comparing these, we get $a_1 = b^2 a^2$ and $A_o^2 = b^2 A^2$. Hence $\varepsilon = A + p(x)\sqrt{f(x)}$ is the fundamental unit of K.

Conversely, if ε is the fundamental unit of K then by Proposition 2.2, K is not of (*) type.

The following lemma will give the existence of infinite numbers of polynomials $f(x)$ which are not of the form $g(x)^2 - a^2$.

LEMMA 2.1. *Suppose that* A, B, C *are polynomials of* $k_o(x)$ *such that deg* $B > 0$ *and* $B \nmid A$. *Then* $B^2 D^2 + AD + C$ *is a square for atmost a finite number of polynomials* D.

Proof. Using $B \nmid A$, we have $A = 2BC_o + C_1$ with $C_1 \neq 0$ and $\deg C_1 < \deg B$. Thus

$$deg\ (A-2BC_o)D = deg(C_1D) < deg(BD) \tag{2.3}$$

for all polynomials $D,\ D \neq 0$. Now write

$$B^2D^2 + AD + C = (BD + C_o)^2 + (A-2BC_o)D + C - C_o^2.$$

Then equation (2.3) gives

$$deg\ ((A-2BC_o)D + C - C_o^2) < deg\ (BD + C_o)$$

for almost all D. Moreover, $(A-2BC_o)D + C - C_o^2 \neq 0$ except for one value of D and hence the required result follows.

In order to give lower bounds for the class number of certain real quadratic extensions of $k_o(x)$ we will prove the following lemmas.

LEMMA 2.2. The Pell's equation

$$A^2 - B^2(g(x)^2 - a) = C,$$

where C is not a square and $deg\ C > 0$, has no solution unless $deg\ C \geq deg\, g(x)$.

Proof. Follows by comparing degrees on both sides.

LEMMA 2.3. Let $K = k_o(x)(\sqrt{f(x)})$ be a real quadratic function field with the fundamental unit $\varepsilon_o = A_o + B_o\sqrt{f(x)}$. Then the Pell's equation

$$A^2 - B^2 f(x) = C,$$

where C is not a square and $deg\ C > 0$, has no solution unless $deg\ C \geq deg\ A_o - 2\ deg\ B_o$.

Proof. We are given that

$$N\ \varepsilon_o = A_o^2 - B_o^2 f(x) = a.$$

Thus $A^2 - B^2 f(x) = C$ has a solution implies that

$$A^2 B_o^2 - B^2 (A_o^2 - a) = C B_o^2$$

is solvable; whence, by lemma 2.2, $deg\,(CB_o^2) \geq deg\,A_o$, i.e. $deg\,C \geq deg\,A_o - 2\,deg\,B_o$.

3. Main Results: Let $p(x)$ be any prime polynomial and consider $f_C(x) = p(x)^2 C^2 + 2aC$ for some polynomial C and a in k_o^*. Then the Pell's equation

$$A^2 - B^2 f_C(x) = a^2$$

has a solution $A = p(x)^2 C^2 + a$, $B = p(x)$.

THEOREM 3.1. Let $p(x)$ be any prime polynomial and consider $f_C(x) = p(x)^2 C^2 + 2aC$ for some polynomial C and a in k_o. Then $K = k_o(x)(\sqrt{f_C(x)})$ is not of (*) type for almost all C and then $\varepsilon_o = (p(x)^2 C^2 + a) + p(x)\sqrt{f_C(x)}$ is its fundamental unit.

Proof. Since $f_C(x) - b$ is not a square, for any element of k_o, for almost all C, by lemma 2.1, we see that K is not of (*) type for almost all C. Then, by Proposition 2.3, the result follows.

THEOREM 3.2. Let $f_C(x) = p(x)^2 C^2 + 2aC$ be as in theorem 3.1. Suppose that $k_o(x)(\sqrt{f_C(x)})$ is not of (*) type and $p(x)$ splits in it. If h is the class number of $k_o(x)(\sqrt{f_C(x)})$ then

$$h \geq \frac{deg\,(p(x)^2 C^2 + a) - 2\,deg\,p(x)}{deg\,p(x)}$$

Proof. Note, first, that

$$deg\,p(x) \leq deg\,(p(x)^2 C^2 + a) - 2\,deg\,p(x),$$

if $deg\,C \geq 1$. The right hand side of the inequality is the boundary condition of lemma 2.3.

By theorem 3.1, $k_o(x)(\sqrt{f_C(x)})$ is not of (*) type implies that $\varepsilon_o = (p(x)^2 C^2 + a) + p(x)\sqrt{f_C(x)}$ is its fundamental unit. Now, by assumption, $(p(x)) = \mathfrak{p}\mathfrak{p}'$ in $k_o(x)(\sqrt{f_C(x)})$. If $\mathfrak{p}$ were principal, the Pell's equation

$$N(\mathfrak{p}) = A^2 - B^2 f_C(x) = c p_o(x)$$

would have a solution for some c in k_o^*.

But then, by lemma 2.3, $deg\, p_o(x) \geq deg\,(p(x)^2 C^2 + a) - 2\, deg\, p(x)$ a contradiction to the choice of C. Thus the order of $\mathfrak{p}$ is greater than one. In particular $\mathfrak{p}^h$ is principal and thus the Pell's equation

$$N(\mathfrak{p}^h) = C p_o(x)^h$$

has a solution. This, by lemma 2.3, implies that

$$deg\; p_o(x)^h \geq deg\,(p(x)^2 C^2 + a) - 2\, deg\, p(x),$$

$$\text{i.e.}\quad h \geq \frac{deg\,(p(x)^2 C^2 + a) - 2\, deg\, p(x)}{deg\, p(x)};$$

hence result follows.

References

1. E. Artin, *Quadratische Körper in Gebiete der höheren Kongruenzen, I,* Math. Z. 19(1924) pp. 153-206.
2. H. Hasse, *Über mehrklassige, aber eingeschlectige reell - quadratische Zahlkörper*, Elemente der Math. 20(1965) pp. 49-59.
3. H. Yokoi, *On the fundamental unit of real quadratic fields with norm 1,* J. Number Theory 2(1970) pp. 106-115.

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