DIFFERENTIABLE PATHS IN TOPOLOGICAL VECTOR SPACES

by

D. F. FINDLEY

In this note, we show that a strong form of the Bolzano-Weierstrass theorem in a topological vector space $E[T]$ is equivalent, for example, to the assertion that there are enough differentiable paths, $x(t)$, with non-trivial tangent vectors, so that a function $f$ defined on $E$ will be sequentially continuous for $T$ if the composites $f(x(t))$ are all continuous. For a large class of locally convex spaces, this property is shown to be equivalent to the statement that the bounded sets of $E[T]$ are finite dimensional. This leads to some very precise results for special cases.

DEFINITIONS. A (continuous) path $x(t) : [0, 1] \to E[T]$ will be called differentiable (directional, tangential) at $t=0$ if the limit as $t$ decreases to 0 of $(x(t) - x(0))/t$ exists in $E[T]$ and is different from 0.

If $E[T]$ is finite dimensional, then a strong form$^1$ of the Bolzano-Weierstrass theorem holds, which says that a bounded sequence $x_n$ in $E[T]$ will have a subsequence $x_{n'}$, which converges in a well-defined direction (cf. Property I (below)). In general, we shall show (cf. (1) and (6)) that this strengthened form of the theo-

---

$^1$ For the "strongest possible" form of the theorem cf. [5].
rem holds if and only if the topological vector space $E[T]$ has one (hence all, cf. (1)) of the properties I-III listed below.

**PROPERTY I.** If $x_n \to x_0$ in $E[T]$, then there exists a path $x(t) : [0,1] \to E[T]$ which is differentiable at $t=0$, where $x(0) = x_0$, and which has the property that for some subsequence $x_{n'}$, of $x_n$ there is a sequence $t_n \to 0$ in $[0,1]$ for which $x(t_{n'}) = x_{n'}$ holds.

**PROPERTY II.** If $x_n \to x_0$ in $E[T]$, there is a subsequence $x_{n'}$ of $x_n$ and a sequence $\alpha_{n'} \to \infty$ of positive numbers such that $\lim_{n} \alpha_{n'}(x_{n'} - x_0)$ exists and is different from 0.

**PROPERTY III.** A map $f$ from an open set $O$ in $E[T]$ to a topological space $S$ is sequentially continuous at $x_0 \in O$ if and only if the composite $f(x(t))$ is continuous at $t=0$, for every path $x(t) : [0,1] \to O$, with $x(0) = x_0$, which is differentiable at $t=0$.

(1) For any topological vector space $E[T]$ the properties I, II and III are equivalent.

Proof. I $\Rightarrow$ II is obvious. (Let $\alpha_{n'} = 1/t_{n'}$.) II $\Rightarrow$ I. Suppose $x_n \to x_0$ is given. Let $x_{n'}$ and $\alpha_{n'}$ be as in II. We define $t_{n'} = 1/\alpha_{n'}$, and we can assume that for each $n'$, $t_{n'}$ belongs to $[0,1]$. We define a path $x(t) : [0,1] \to E[T]$ as follows: We set $x(0) = x_0$ and $x(t) = x_1$, if $t \in [t_{n'}, 1]$. Otherwise, for $t = \alpha t_{n'} + (1-\alpha) t_{(n+1)}$, $0 \leq \alpha \leq 1$ we define

$$x(t) = \alpha x_{n'} + (1-\alpha) x_{(n+1)}'.$$

Since for $t \in [t_{(n+1)'}, t_{n'}]$ we have

$$\frac{x(t) - x_0}{t} = \frac{\alpha(x_{n'} - x_0) + (1-\alpha)(x_{(n+1)} - x_0)}{\alpha t_{n'} + (1-\alpha) t_{(n+1)'}}$$

248
it is clear that
\[
\lim_{t \to 0} \frac{(x(t) - x_0)}{t} = \lim_{n' \to n'} (x_{n'} - x_0).
\]

Hence \(x(t)\) has the property required in \(I\). The proof of \(I \Rightarrow III\) is straightforward. We show \(I \Rightarrow III\).

Suppose \(x_n\) converges to \(x_0\) in such a way that no subsequence \(x_{n'}\) lies on path \(x(t)\), with \(x(0) = x_0\), which is differentiable at \(t = 0\). We define \(f(x)\) to be 1 if \(x \in \{x_n : n=1, 2, \ldots\}\) and 0 otherwise. Then \(\lim f(x_n) = 1 \neq 0 = f(x_0)\), but for any path \(x(t)\), differentiable at \(t = 0\) and such that \(x(0) = x_0\), we have \(\lim_{t \to 0} f(x(t)) = 0 = f(x_0)\), by the hypothesis on \(x_n\) and by the definition of \(f\).

Thus \(III\) is contradicted.

There are infinite dimensional topological vector spaces possessing properties \(I-III\). For any infinite set \(A\) let \(\omega_A\) denote the set of all complex-valued functions on \(A\) and let \(\phi_A\) denote the subspace of functions having finite support.

In the dual system \(\langle \phi_A, \omega_A \rangle\) formed in the usual way, all \(T_s (\omega_A)\)-bounded sets have finite dimensional span. This is also the essential property of both spaces \(E\) in \(E^*\) in the dual systems \(\langle E, E^* \rangle\) of the very interesting class of spaces studied by Y. Komura and Amemiya (cf. [1]). In such spaces \(I-III\) clearly hold. These examples are quite typical as the next theorems show.

First, we define:

**PROPERTY IV.** If \(x_n\) is a bounded sequence in \(E[T]\), then span \(\{x_n : n = 1, 2, \ldots\}\) is finite dimensional.

(2) If \(E[T]\) is a locally convex topological vector space for which there exists a weaker topology than \(T\) which is metrizable, then \(IV\) is equivalent to \(I-III\).
Proof. \( IV \Rightarrow II \) is elementary. The reverse implication follows from (3) below, which shows that \(- IV\) implies a strong form of \(- II\).

(3) Let \( E[T] \) satisfy the hypothesis of (2) and suppose that \( z_n \) is a bounded sequence of linearly independent elements in \( E \). Then there is a sequence \( x_n \) in \( F = \text{span} \{ z_n : n = 1, 2, \ldots \} \) which converges to 0 and which has the following property:

If for some scalar sequence \( \alpha_n \) the sequence \( \alpha_n x_n \) has a \( Ts(E') \)-adherent point \( x \) in \( E \), then \( x = 0 \). \((E' \text{ denotes the dual of } E[T].)\)

Proof. Let \( F \) denote the weak closure of \( F \) in \( E \). With respect to the induced weaker metrizable locally convex topology \( \overline{F} \) is a separable metric space, so by ([3] 21, 3, (5)) its dual \( F' \) is weakly separable. Hence, there is a linearly independent sequence \( \psi_m \) in \( F' \) with the property that if for some \( x \in F \) we have \( \psi_m(x) = 0 \) for all \( m \), then \( x = 0 \).

With the aid of the Hahn-Banach theorem applied in the dual system \( <F', F> \), we can obtain sequences \( \phi_m \) in \( F' \) and \( y_n \) in \( F \) such that

\[
\begin{align*}
\text{(*)&} & \quad \text{span} \{ \phi_1, \ldots, \phi_m \} = \text{span} \{ \psi_1, \ldots, \psi_m \} \quad \text{for all } m, \quad \text{and} \\
\text{(**)} & \quad \phi_m(y_n) \neq 0 \quad \text{if and only if} \quad m = n; \quad m, n = 1, 2, \ldots.
\end{align*}
\]

Since each \( y_n \) is a finite linear combination of elements from the \( T \)-bounded set \( \{ z_n : n = 1, 2, \ldots \} \), we can find a sequence of non-zero scalars \( \beta_n \) such that \( x_n = \beta_n y_n \) is \( T \)-convergent to 0. Let \( \alpha_n \) be any sequence of scalars and suppose that \( x \) is a \( Ts(E') \)-adherent point of \( \alpha_n x_n \). For any fixed \( m \), it follows that there is a subsequence \( \alpha_n', x_n' \) such that

\[
\phi_m(x) = \lim_{n' \to m} \phi_m(\alpha_n' x_n') = \lim_{n' \to m} \alpha_n' \beta_n' \phi_m(y_{n'}) = 0
\]
But it follows from (*) that $\phi_m$ also separates the points of $F$, so we must have $x = 0$. Thus the sequence $x_n$ has the sought after property.

If $E[T\ell]$ is itself locally convex and metrizable, with metric $\rho(x,y)$, we can say more. For if $y_n$ is any linearly independent sequence in $E$, then we can choose non-zero scalars $\delta_n$ so that $\rho(\delta_n y_n, 0) < 1/n$. The sequence $z_n = \delta_n y_n$ is then bounded so that (3) applies. Hence, by (2):

(4) A locally convex and metrizable space $E[T\ell]$ has one of the properties I-IV if and only if $E$ is finite dimensional.

In a similar vein, it follows from (4) and ([3, 19, 5, (5) and 22, 6, (4); cf. also [4]) that:

(5) In a quasi-complete (LF)-space $E[T\ell]$, the properties I-IV are each equivalent to the assertion that $E[T\ell]$ is of the form $\phi_A[T_S(\omega_A)]$ (cf. above (2)) for some countable set $A$.

There are some simple observations we can use to say something about the properties I-III in a general topological vector space $E[T\ell]$.

(6) If $E[T\ell]$ has the properties I-III, then every bounded sequence in $E[T\ell]$ has a convergent subsequence.

Proof. Suppose that II holds and let $x_n$ be a bounded sequence. Since $x_n/n$ converges to $0$, there must exist, by II, a subsequence $x_{n^*}$, and a scalar sequence $\alpha_{n^*}$, such that $(\alpha_{n^*}/n^*)x_{n^*}$ converges to some $x \neq 0$. Because $x$ is not 0, no subsequence of $\alpha_{n^*}/n^*$ is convergent to 0. Hence $n^*/\alpha_{n^*}$ is bounded and there is a subsequence $n^{**}/\alpha_{n^{**}}$ of $n^*/\alpha_{n^*}$ which is convergent, say to $\alpha$. This implies that $x_{n^{**}} = (n^{**}/\alpha_{n^{**}})(\alpha_{n^{**}}/n^{**})x_{n^{**}}$ converges to $\alpha x$.

(7) If each neighborhood of 0 of the metrizable space $E[T\ell]$ contains a ray
\{a \in \mathbb{R} : a > 0 \} \ (x \neq 0)$, then there is a sequence $x_n \to 0$ in $E[T]$ such that for any choice of scalars $\alpha_n > 0$, the sequence $\alpha_n x_n$ converges to 0. Hence $E[T]$ does not have properties I-IV.

**Proof.** Let $U_1 \supseteq \ldots \supseteq U_n \supseteq \ldots$ be a fundamental system of neighborhoods of 0. Let the sequence $x_n$ be so chosen that for each $n$, $\{a x_n : a > 0 \} \subseteq U_n \setminus \{0\}$. Then $\alpha_n x_n \in U_n$ for all $n \geq n_0$, for any $n_0$ and for any choice of non-negative scalars $\alpha_n$.

**Remark:** We can weaken what we have called Properties I-III by allowing as differentiable paths $x(t)$ for which $x'(0)$ may be 0. It follows from Lemma 3.3 (p. 99) of [2] that every topological vector space on which continuity and sequential continuity coincide has these weaker properties (which are equivalent by the proof of (1)).

**REFERENCES**


Department of Mathematics
University of Cincinnati
Cincinnati, Ohio, 45221, E.U.A.

(Recibido en febrero de 1974).