Revista Colombiana de Matemáticas
Volumen VIII (1974), págs. 247 - 252

DIFFERENTIABLE PATHS IN TOPOLOGICAL VECTOR SPACES

by

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In this note, we show that a strong form of the Bolzano-Weierstrass theorem in a topological vector space $E[T]$ is equivalent, for example, to the assertion that there are enough differentiable paths, $x(t)$, with non-trivial tangent vectors, so that a function f defined on E will be sequentially continuous for T if the composites $f(x(t))$ are all continuous. For a large class of locally convex spaces, this property is shown to be equivalent to the statement that the bounded sets of $E[T]$ are finite dimensional. This leads to some very precise results for special cases.

DEFINITIONS. A (continuous) path $x(t):[0,1]\to E[T]$ will be called *differentiable (directional, tangential)* at $t=0$ if the limit as t decreases to 0 of $(x(t)-x(0))/t$ exists in $E[T]$ and is *different from* 0.

If $E[T]$ is finite dimensional, then a strong form[1] of the Bolzano-Weierstrass theorem holds, which says that a bounded sequence x_n in $E[T]$ will have a subsequence $x_{n'}$, which converges in a well-defined direction (cf. Property I (below)). In general, we shall show (cf. (1) and (6)) that this strengthened form of the theo-

1 For the "strongest possible" form of the theorem cf. [5].

rem holds if and only if the topological vector space $E[T]$ has one (hence all, cf. (1)) of the properties I-III listed below.

PROPERTY I. If $x_n \to x_o$ *in* $E[T]$, *then there exists a path* $x(t) : [0,1] \to E[T]$ *which is differentiable at* $t=0$, *where* $x(0) = x_o$, *and which has the property that for some subsequence* $x_{n'}$ *of* x_n *there is a sequence* $t_{n'} \downarrow 0$ *in* $[0,1]$ *for which* $x(t_{n'}) = x_{n'}$ *holds.*

PROPERTY II. If $x_n \to x_o$ *in* $E[T]$, *there is a subsequence* $x_{n'}$ *of* x_n *and a sequence* $\alpha_{n'} \uparrow \infty$ *of positive numbers such that* $\lim_n \alpha_{n'}(x_{n'} - x_o)$ *exists and is different from* 0.

PROPERTY III. A map f *from an open set* O *in* $E[T]$ *to a topological space* S *is sequentially continuous at* $x_o \in O$ *if and only if the composite* $f(x(t))$ *is continuous at* $t=0$, *for every path* $x(t) : [0,1] \to O$, *with* $x(0) = x_o$, *which is differentiable at* $t=0$.

(1) For any topological vector space $E[T]$ *the properties I, II and III are equivalent.*

Proof. I => II is obvious. (Let $\alpha_{n'} = 1/t_{n'}$). II => I. Suppose $x_n \to x_o$ is given. Let $x_{n'}$ and $\alpha_{n'}$ be as in II. We define $t_{n'} = 1/\alpha_{n'}$ and we can assume that for each n', $t_{n'}$ belongs to $[0,1]$. We define a path $x(t) : [0,1] \to E[T]$ as follows : We set $x(0) = x_o$ and $x(t) = x_{1'}$ if $t \in [t_{1'}, 1]$. Otherwise, for $t = \alpha t_{n'} + (1-\alpha) t_{(n+1)'}$, $(0 \leq \alpha \leq 1)$ we define

$$x(t) = \alpha x_{n'} + (1-\alpha) x_{(n+1)'} .$$

Since for $t \in [t_{(n+1)'}, t_{n'}]$ we have

$$\frac{x(t) - x_o}{t} = \frac{\alpha(x_{n'} - x_o) + (1-\alpha)(x_{(n+1)'} - x_o)}{\alpha t_{n'} + (1-\alpha) t_{(n+1)'}}$$

it is clear that

$$\lim_{t \downarrow 0} \frac{(x(t) - x_o)}{t} = \lim_{n'} \alpha_{n'} (x_{n'} - x_o) .$$

Hence $x(t)$ has the property required in I . The proof of I => III is straightforeward. We show I => III .

Suppose x_n converges to x_o in such a way that no subsequence $x_{n'}$ lies on path $x(t)$, with $x(0) = x_o$, which is differentiable at $t = 0$. We define $f(x)$ to be 1 if $x \in \{x_n : n = 1, 2, \ldots\}$ and 0 otherwise. Then $\lim_n f(x_n) = 1 \neq 0 = f(x_o)$, but for any path $x(t)$, differentiable at $t = 0$ and such that $x(0) = x_o$, we have $\lim_{t\, o} f(x(t)) = 0 = f(x_o)$, by the hypothesis on x_n and by the definition of f. Thus III is contradicted.

There are infinite dimensional topological vector spaces possessing properties I-III. For any infinite set A let ω_A denote the set of all complex-valued functions on A and let ϕ_A denote the subspace of functions having finite support . In the dual system $\langle \phi_A, \omega_A \rangle$ formed in the usual way, all $T_s(\omega_A)$-bounded sets have finite dimensional span. This is also the essential property of both spaces E in E' in the dual systems $\langle E, E' \rangle$ of the very interesting class of spaces studied by Y . Komura and Amemiya (cf. [1]) . In such spaces I-III clearly hold. These examples are quite typical as the next theorems show.

First, we define :

PROPERTY IV. If x_n *is a bounded sequence in* $E[T]$, *then span* $\{x_n : n = 1, 2, \ldots\}$ *is finite dimensional.*

(2) If $E[T]$ *is a locally convex topological vector space for which there exists a weaker topology than* T *which is metrizable, then IV is equivalent to I-III.*

Proof. IV => II is elementary. The reverse implication follows from (3) below, which shows that ~ IV implies a strong form of ~II.

(3) Let $E[T]$ satisfy the hypothesis of (2) and suppose that z_n is a bounded sequence of linearly independent elements in E. Then there is a sequence x_n in $F = span\{z_n : n = 1, 2, \ldots\}$ which converges to 0 and which has the following property:

If for some scalar sequence α_n the sequence $\alpha_n x_n$ has a $T_s(E')$-adherent point x in E, then $x = 0$. (E' denotes the dual of $E[T]$.)

Proof. Let $\overline{F}$ denote the weak closure of F in E. With respect to the induced weaker metrizable locally convex topology $\overline{F}$ is a separable metric space, so by ([3] 21, 3. (5)) its dual F' is weakly separable. Hence, there is a linearly independent sequence ψ_m in F' with the property that if for some $x \in \overline{F}$ we have $\psi_m(x) = 0$ for all m, then $x = 0$.

With the aid of the Hahn-Banach theorem applied in the dual system $\langle F', F \rangle$, we can obtain sequences ϕ_m in F' and y_n in F such that

(*) $span\{\phi_1, \ldots, \phi_m\} = span\{\psi_1, \ldots, \psi_m\}$ for all m, and

(**) $\phi_m(y_n) \neq 0$ if and only if $m = n$; $m, n = 1, 2, \ldots$.

Since each y_n is a finite linear combination of elements from the T-bounded set $\{z_n : n = 1, 2, \ldots\}$, we can find a sequence of non-zero scalars β_n such that $x_n = \beta_n y_n$ is T-convergent to 0. Let α_n be any sequence of scalars and suppose that x is a $T_s(E')$-adherent point of $\alpha_n x_n$. For any fixed m, it follows that there is a subsequence $\alpha_{n'} x_{n'}$ such that

$$\phi_m(x) = \lim_{n' > m} \phi_m(\alpha_{n'} x_{n'}) = \lim_{n' > m} \alpha_{n'} \beta_{n'} \phi_m(y_{n'}) = 0$$

But it follows from (*) that ϕ_m also separates the points of $\overline{F}$, so we must have $x=0$. Thus the sequence x_n has the sought after property.

If $E[T]$ is itself locally convex and metrizable, with metric $\rho(x,y)$, we can say more. For if y_n is any linearly independent sequence in E, then we can choose non-zero scalars δ_n so that $\rho(\delta_n y_n, 0) < 1/n$. The sequence $z_n = \delta_n y_n$ is then bounded so that (3) applies. Hence, by (2) ;

(4) A locally convex and metrizable space $E[T]$ has one of the properties I-IV if and only if E is finite dimensional.

In a similar vein, it follows from (4) and ([3] 19,5. (5) and 22,6 .(4) ; cf. also [4]) that :

(5) In a quasi-complete (LF)-space $E[T]$, the properties I-IV are each equivalent to the assertion that $E[T]$ is of the form $\phi_A[T_s(\omega_A)]$ (c.f. above (2)) for some countable set A.

There are some simple observations we can use to say something about the properties I-III in a general topological vector space $E[T]$.

(6) If $E[T]$ has the properties I-III, then every bounded sequence in $E[T]$ has a convergent subsequence.

Proof. Suppose that II holds and let x_n be a bounded sequence. Since x_n/n converges to 0, there must exist, by II, a subsequence $x_{n'}$, and a scalar sequence $\alpha_{n'}$, such that $(\alpha_{n'}/n')x_{n'}$ converges to some $x \neq 0$. Because x is not 0, no subsequence of $\alpha_{n'}/n'$ is convergent to 0. Hence $n'/\alpha_{n'}$ is bounded and there is a subsequence $n''/\alpha_{n''}$ of $n'/\alpha_{n'}$ which is convergent, say to α. This implies that $x_{n''} = (n''/\alpha_{n''})(\alpha_{n''}/n'')x_{n''}$ converges to αx.

(7) If each neighborhood of 0 of the metrizable space $E[T]$ contains a ray

$\{\alpha x : \alpha > 0\}$ $(x \neq 0)$, then there is a sequence $x_n \to 0$ in $E[T]$ such that for any choice of scalars $\alpha_n \geq 0$, the sequence $\alpha_n x_n$ converges to 0. Hence $E[T]$ does not have properties I-IV.

Proof. **Let $U_1 \supseteq \ldots \supseteq U_n \supseteq \ldots$ be a fundamental system of neighborhoods of 0. Let the sequence x_n be so chosen that for each n, $\{\alpha x_n : \alpha > 0\} \subseteq U_n \setminus \{0\}$. Then $\alpha_n x_n \in U_{n_o}$ for all $n \geq n_o$, for any n_o and for any choice of non-negative scalars α_n.**

Remark : **We can weaken what we have called Properties I-III by allowing as differentiable, paths $x(t)$ for which $x'(0)$ may be 0. It follows from Lemma 3.3 (p. 99) of [2] that every topological vector space on which continuity and sequential continuity coincide has these weaker properties (which are equivalent by the proof of (1)).**

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(Recibido en febrero de 1974).