Revista Colombiana de Matemáticas
Volumen IX (1975), págs. 1 - 7

NOTE ON THE TRUNCATED RAYLEIGH VARIATE

by

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SUMMARY

In this brief note, the truncated Rayleigh variate is considered. If the truncation point is assumed to be known, maximum likelihood estimate for the parameter is derived and maximum likelihood ratio test for the simple null hypothesis with simple alternative is developed for only one observation. Power of this test is tabulated for usual significance levels.

Introduction. If X is a positive random variable with density function $f(x;\theta)$, $x, \theta > 0$, then it is known that if x_T is a truncation point that is, X_T is defined for $x > x_T$ -the density function of the truncated variable is :

$$X_T : f_T(x;\theta) = \frac{1}{1-F(x_T;\theta)} \cdot f(x;\theta) \quad \text{for } x > x_T,\ \theta > 0, \quad (1)$$

where $F(x_T;\theta)$ is the distribution function of X, calculated in the truncation point.

Consider now a Rayleigh distributed random variable :

$$X_R : f(x;\theta) = 2\theta x \exp(-\theta x^2) \qquad x > 0, \theta > 0 \tag{2}$$

Then $F(x;\theta) = 1-\exp(-\theta x^2)$. Let x_T be a known truncation point. Therefore $1-F(x_T;\theta) = \exp(-\theta x_T^2)$ and then :

$$X_{TR} : f_T(x;\theta) = 2\theta x \exp\{-\theta(x^2 - x_T^2)\} \quad , x > x_T , \theta > 0 . \tag{3}$$

2. *Estimation of the parameter.* Let $x_1, x_2, \ldots, x_n$ be an independent sample on X_{TR} with $x_i > x_T$ for every $i = 1, 2, \ldots, n$. Therefore the likelihood function is :

$$L = \prod_{i=1}^{n} f_T(x_i;\theta) = 2^n \theta^n \prod_{i=1}^{n} x_i \exp\{-\theta \sum_{i=1}^{n} (x_i^2 - x_T^2)\} \tag{4}$$

From the above relation we get :

$$\ln L = n \ln\theta - \theta \sum_{i=1}^{n} x_i^2 + \theta n x_T^2 + n \ln 2 + \sum_{i=1}^{n} \ln x_i , \tag{5}$$

$$\frac{\partial \ln L}{\partial \hat{\theta}} = n\hat{\theta}^{-1} - \sum_{i=1}^{n} x_i^2 + n x_T^2 = 0 , \tag{6}$$

with the solution :

$$\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i^2 - n x_T^2} . \tag{7}$$

It is obvious that $\sum_{i=1}^{n} x_i^2 > n x_T^2$. In the same way, the m. l.e. of $1/\theta$ is:

$$\widehat{\left(\frac{1}{\theta}\right)}_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - x_T^2 \tag{8}$$

It is to note that for the truncated case, $\widehat{\left(\frac{1}{\theta}\right)}_{ML}$ does not posses unbiasedness property. Indeed, straightforward computations will provide :

$$E(X_{TR}) = \frac{1}{2}\,\theta^{-\frac{1}{2}}\sqrt{\pi} + x_T\, exp(-\theta x_T^2) - \Phi_L(x_T\sqrt{2\pi})\,; \tag{9}$$

$$E(X_{TR}^2) = \theta^{-1}\, exp(-\theta x_T^2) + x_T^2\, exp(-\theta x_T^2)\,. \tag{10}$$

Here :

$$\Phi_L(a) = \frac{1}{\sqrt{2\pi}}\int_o^a exp(-t^2/2)\,dt\,. \tag{11}$$

It follows that :

$$E\left[\widehat{\left(\frac{1}{\theta}\right)}_{ML}\right] = \frac{1}{\theta}\, exp(-\theta x_T^2) + (\, exp(-\theta x_T^2) - 1)\, x_T^2 \tag{12}$$

If $x_T = 0$ **we obtain that m.l.e. of** $1/\theta$ **is unbiased.**

3. Testing a simple hypothesis. **In this paragraph we shall test statistical hypothesis** $H_o = \theta = \theta_o$ **against** $H_1 : \theta = \theta_1 \quad (\theta_o < \theta_1)$ (13)

We shall construct likelihood ratio test for only one observation. We have :

$$r(x) = \frac{f_T(x\,;\theta_1)}{f_T(x\,;\theta_o)} = exp\{-(\theta_1-\theta_o)x^2\}\, exp\{(\theta_1-\theta_o)\,x_T^2\} \tag{14}$$

The behaviour of the likelihood ratio is showed in the figure below

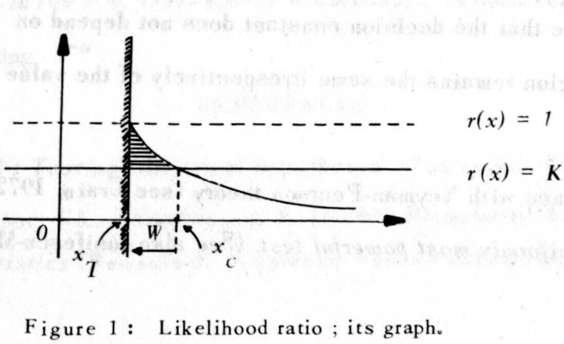

Figure 1 : Likelihood ratio ; its graph.

It follows that conditions from Neyman-Pearson lemma are satisfied if the critical region is defined as :

$$W \equiv (x_T ; x_c) \tag{15}$$

The value x_c is determined such that the significance level of the test be a preassigned value, say ε. We have :

$$\int_{x_T}^{x_c} f_T(x ; \theta_o)\, dx = \varepsilon \quad \text{or} \quad 1 - exp\{-\theta (x^2 - x_T^2)\} = \varepsilon \tag{16}$$

From the above relation we obtain immediately :

$$x_c = \left[x_T^2 - ln\,(1-\varepsilon)^{\frac{1}{\theta_o}} \right]^{\frac{1}{2}} \tag{17}$$

The value K is determined from the relation $r(x) = K$. We get by taking logarithms :

$$x_c = \left[x_T^2 - ln\, K^{1/(\theta_1 - \theta_o)} \right]^{\frac{1}{2}} \tag{18}$$

The decision is therefore :

a) if $x \geq [\, x_T^2 - ln\,(1-\varepsilon)^{1/\theta_o}]^{\frac{1}{2}}$ then accept H_o ; (19)

b) if $x < [\, x_T^2 - ln\,(1-\varepsilon)^{1/\theta_o}]^{\frac{1}{2}}$ then accept H_1 . (20)

It is to note that the decision constant does not depend on θ_1 ; therefore the critical region remains the same irrespectively of the value of θ_1 greater than θ_o .

In accordance with Neyman-Pearson theory (see Craiu, 1972) the test constructed is a *uniformly most powerful test*. (See also Iosifescu-Mihoc-Theodorescu, 1966)

The power of the test is defined as :

$$\pi(\theta_1) = \int_{x_T}^{x_c} f_T(x;\theta_1)\,dx ; \qquad (21)$$

that is

$$\pi(\theta_1) = 1-exp\{-\theta_1(x^2-x_T^2)\} = 1-exp\{-\theta_1[-\frac{1}{\theta_o}\ ln\,(1-\varepsilon)]\} = 1-exp[ln\,(1-\varepsilon)^{\theta_1/\theta_o}] =$$

$$= 1-(1-\varepsilon)^{\theta_1/\theta_o} \qquad (22)$$

Denote $\frac{\theta_1}{\theta_o} = \delta$. Hence :

$$\pi(\delta) = 1-(1-\varepsilon)^{\delta} \qquad (23)$$

(taking $\theta_1 = \theta_o$ we shall have $\pi(\delta) = \varepsilon$ that is significance level) .

It is important to note that the *power of the test does not involve the truncation point.*

The table at the end gives power of the test $\pi(\delta)$ for $\varepsilon = 0.05\ ;\ 0.01$ and $\delta = 1\,(0.20)\,8(1)\,10$.

Computations have been made in the form :

$$\pi(\delta) = 1-e^{\delta\,ln(1-\varepsilon)} = 1-e^{\delta(-0.0513)} ; \ (\varepsilon = 0.05)$$

$(ln\ 0.95 = ln\,95 - ln\,100 \cong 4.5538-4.6051 = -0.0513)$. A desk computer has been used for computing e^{-u} .

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(Recibido en julio de 1974) .

δ	$\pi(\delta)$		δ	$\pi(\delta)$	
	$\varepsilon = 0.05$	$\varepsilon = 0.01$		$\varepsilon = 0.05$	$\varepsilon = 0.01$
1.00	0.05000	0.01000	4.80	0.21979	0.04708
1.20	0.05965	0.01195	5.00	0.22624	0.04901
1.40	0.06926	0.01394	5.20	0.23480	0.05192
1.60	0.07821	0.01592	5.40	0.24194	0.05281
1.80	0.08815	0.01790	5.60	0.24965	0.05472
2.00	0.09750	0.01990	5.80	0.25731	0.05662
2.20	0.10668	0.02196	6.00	0.26492	0.05852
2.40	0.11714	0.02380	6.20	0.27245	0.06042
2.60	0.12621	0.02578	6.40	0.27988	0.06228
2.80	0.13507	0.02775	6.60	0.29430	0.06414
3.00	0.14394	0.02970	6.80	0.29449	0.06604
3.20	0.15263	0.03165	7.00	0.30169	0.06793
3.40	0.16128	0.03358	7.20	0.30883	0.06980
3.60	0.16990	0.03552	7.40	0.31588	0.07169
3.80	0.17832	0.03747	7.60	0.32287	0.07354
4.00	0.18674	0.03941	7.80	0.32978	0.07541
4.20	0.19545	0.04137	8.00	0.33663	0.07723
4.40	0.20404	0.04327	9.00	0.36979	0.08651
4.60	0.21221	0.04514	10.00	0.40132	0.09562

Table 1 : Power of the L.R.T. for one observation

($\varepsilon = 0.05$; 0.01 ; $\delta = 1\,(0.20)\,8(1)\,10$) .