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SOME NOTES ON RAYLEIGH DISTRIBUTION

by

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SUMMARY

In this paper, we emphasize some aspects concerning the Rayleigh distribution. After a brief presentation of the literature devoted to this variable, we give a characterization of the Rayleigh distribution by using a Gercbach idea (1967). Then, some properties are put into ligth and in the latter part of the work sequential tests are constructed in order to test a simple and a compound hypothesis concerning a Rayleigh parameter.

1. ***Introduction.*** **It is well-known that a randon variable T obeys a Rayleigh law, if its density function has the form :**

$$T : f(t;\lambda) = 2\,\lambda^{-2}\, t\, exp\,\{-\lambda^{-2}\, t^2\}\,,\quad t \geq 0\,,\quad \lambda > 0. \qquad (1)$$

It can be easily obtained by taking in the expression of the Weibull law :

$$T_w : f(t;\gamma\,,p) = \gamma\, p\, t^{p-1}\ exp\,\{-\gamma\, t^p\}\,,\quad t \geq 0,\ \gamma,\ p > 0, \qquad (2)$$

respectively, $\gamma = \lambda^{-2}$, and $p = 2$.

Bruscantini (1968) enlists some important occurences of the Rayleigh law in various practical situations, namely :

- in naval research, to simulate the sea waves ;

- in telecommunications, to describe the signal fluctuations due to multipath effects in the line -of - sight links ;

- in bombing problems, to describe the distribution of distances from target to the actual impact points.

In some cases, Rayleigh law appears naturally : let us consider a system built up by n pairs of elements, connected in series. In every pair, each element is functionning and a failure occurs if at least a pair is failed. Suppose that the reliability of each element obeys an exponential law and let λ_k be the hazard rate of the k^{th} pair. If $\theta = \left(\sum_{k=1}^{n} \lambda_k^2 \right)^{\frac{1}{2}}$ is fixed and $\lim_n \sum_{k=1}^{n} \lambda_k^3 = 0$, then the asymptotic reliability function of the system is given by :

$$AsP(t) = \lim_n P(t) = exp\,\{-\theta^2 t^2\}\ , \quad t \geq 0\ , \ \theta > 0\ , \tag{3}$$

that is a Rayleigh law (see Gnedenko-Belyaev-Solovyev, 1965).

It is to note also that the Rayleigh law can be obtained not only from the Weibull distribution but even as a peculiar case of other important probability laws .

Let us remember the "chi-distribution" and so-called "generalized Rayleigh variable" whose density is :

$$f(t;\theta,k) = \frac{2\,\theta^{k+1}}{\Gamma\,(k+1)}\, t^{2k+1} exp\{-\theta\, t^2\}\,, \quad t \geq 0, \theta, k > 0, \tag{4}$$

where, if we take $\theta = \lambda^{-2}$ and $k=0$, simple Rayleigh law is obtained (see for details, Vodă, 1972) .

Finally, let us note that if $M(X,Y)$ is a point in a rectangular coordinate - system, where X and Y are random variables each of class $N(0\ ;\sigma^2)$, then the distance $d = (X^2 + Y^2)^{\frac{1}{2}}$ is distributed as a random variable which obeys

Rayleigh law . (See Freeman, 1963 p. 155) .

In various statistical textbooks (see, for instance, Maisel, 1971) the Rayleigh density function appears as below :

$$f(x ; \alpha) = \alpha^{-2} x \exp\{-x^2 / 2\alpha^2\}, x \geq 0 , \alpha > 0 \tag{5}$$

The specific literature devoted to the Rayleigh distribution is not so very wide but most of the properties of this distribution can be derived from properties of the Weibull law .

We shall mention the works of Geldston (1962) who has investigated the variable $Y = k \log (1 + bX)$ where X has the density (1), Krysicki (1963) who has applied method of moments in estimating parameters of a mixture of two Rayleigh distributions (in this connection see also Siddiqui and Weiss , 1963) and the paper of Archer (1967) who has presented some properties of Rayleigh distributed random variables and of their sums and products .

2. *A characterization of Rayleigh distribution.* Let us consider the following function :

$$F(t; \lambda , \xi) = 1 - \exp(-t^2 / \lambda^2) \cdot P(t; \xi) \quad \text{for } t \geq 0, \lambda, \xi > 0, \tag{6}$$

where

$$P(0 ; \xi) = 1 \quad \text{and} \quad P(\infty ; \xi) = 1 \tag{7}$$

It is clear that in this way , $F(t; \lambda, \xi)$ may be interpreted as a distribution function and more than that, $P(t; \xi)$ may be not necessarily a reliability function (see for instance Gercbach, 1967) .

If, in particular, there exists a value $\xi = \xi_o$ such that for every $t \geq 0$, $P(t ; \xi_o) = 1$, then the Rayleigh distribution function is rediscovered .

Let $P(t; \xi)$ be of the following type :

$$P(t\,;\,\xi) = \exp\{\alpha(t)\,r(\xi) + \beta(t) + s(\xi)\},\ t \geq 0,\ \xi > 0 \qquad (8)$$

and preserving the properties (7) .

In this situation one can give the following characterization of the Rayleigh law:

LEMMA 1 : Let $T_1, T_2, \ldots, T_n$ *be independent and identically distributed random variables. Then the distribution of* $\min(T_1, T_2, \ldots, T_n)$ *is of type (6) if and only if the* T_i*'s distributions are of the same type .*

Proof: **Let** $T_{(1)} < T_{(2)} < \cdots < T_{(n)}$ **be the ordered variables** $T_1, \ldots, T_n$, **each with cumulative distribution function given by (6) . Then , we can write :**

$$F_o(t) = \operatorname{Prob}\{T_{(1)} < t\} = 1 - \operatorname{Prob}\{T_{(1)} > t\} = 1 - \operatorname{Prob}\{\text{all } T_i > t\} = 1 - [1 - F(t;\lambda,\xi)]^n =$$

$$= 1 - \exp(-n t^2/\lambda^2)\,[P(t;\xi)]^n = \exp(-n t^2/\lambda^2)\,.\,\exp\{n\,\alpha(t)\,r(\xi) + n\,\beta(t) + n s(\xi)\} \qquad (9)$$

If we denote $n\lambda^{-2} = \theta^{-2}$, $\theta > 0$, $\alpha(t) = \tilde{\alpha}(t)$, $n r(\xi) = \tilde{r}(\xi)$ $n\beta(t) = \tilde{\beta}(t)$ **and** $n s(\xi) = \tilde{s}(\xi)$, **we obtain :**

$$F_o(t) = 1 - \exp(-t^2/\theta^2)\ \tilde{P}(t\,;\,\xi) \quad , \qquad (10)$$

that is a distribution of the type (6) .

Lemma is proved .

3. Estimation problems. **The expected-value and the variance of a Rayleigh variate are :**

$$E(T) = \frac{\sqrt{\pi}}{2}\,\lambda \quad \text{and} \quad Var(T) = \left(1 - \frac{\pi}{4}\right)\lambda^2 \qquad (11)$$

The square of a Rayleigh variate T^2 **obeys an exponential law :**

$$T^2 : f_{T^2}(t\,;\,\lambda) = \frac{1}{\lambda^2}\exp(-t/\lambda^2),\ t \geq 0,\ \lambda > 0 \qquad (12)$$

As a consequence, the maximum likelihood estimate of λ^2 **, that is :**

$$(\hat{\lambda}^2)_{ML} = \frac{1}{n} \sum_{i=1}^{n} t_i^2 \quad , \qquad (13)$$

$t_1, t_2, \ldots, t_n$ being and independent random sample drawn from T, is unbiased and with minimum variance .

On the other hand, the statistic :

$$h(t) = \frac{\sqrt{\pi}}{2} \, t \qquad (14)$$

is an unbiased estimate for λ with respect to the loss-function :

$$W[\lambda \; ; h(t)] = \frac{1}{\lambda^2} [\, h(t) - \lambda \,]^2 \qquad (15)$$

The associated risk is :

$$R_{h(t)}(\lambda) = E\{ W[\lambda; h(t)] \} = 1 - \frac{\pi}{4} \qquad (16)$$

and the variance :

$$Var(h(t)) = E[\, h(t) - E(h(t))\,]^2 = \frac{\pi}{4}\left(1 - \frac{\pi}{4}\right)\lambda^2 \qquad (17)$$

In the following we shall derive the maximum likelihood estimate for λ^2 in the case of a special incomplete sample.

Therefore, suppose that from n observations we have the possibility to record only :

$$t_{(m)} < t_{(m+1)} < \ldots < t_{(m+k)} \qquad (18)$$

where $1 < m < m+k < n$.

The lack of the first m-1 observations may be interpreted as "infant mortality" of certain cutting-tools and the absence of the last observations may be due to the technological restrictions .

Hence, the likelihood function in this case has the form :

$$L = [\ 1-F(t_{(m+k)};\lambda)\ ]^{n-m-k} \cdot [\ F(t_{(m)};\lambda)]^{m-1} \prod_{i=m}^{m+k} f(t_{(i)};\lambda) \tag{19}$$

Hence :

$$\ln L = (n-m-k)\ \ln [\ 1-F(t_{(m+k)};\lambda)\ ] + (m-1) \ln F(t_{(m)};\lambda) + \sum_{i=m}^{m+k} \ln f(t_{(i)};\lambda) \tag{20}$$

or in our particular case :

$$\ln L = (n-m-k) \cdot \left(- \frac{t^2_{(m+k)}}{\lambda^2} \right) + (m-1) \ln [1-\exp\{-t^2_{(m)}/\lambda^2\}] + \sum_{i=1}^{m+k} \left(\ln \frac{2}{\lambda^2} + \ln t_{(i)} - \frac{t^2_{(i)}}{\lambda^2} \right) \tag{21}$$

$$\frac{\partial \ln L}{\partial (\lambda^2)} = (n-m-k) \cdot \left(\frac{t^2_{(m+k)}}{\lambda^4} \right) + (m-1) \frac{-\frac{1}{\lambda^4} t^2_{(m)} \exp\{-t^2_{(m)}/\lambda^2\}}{1-\exp\{-t^2_{(m)}/\hat{\lambda}^2\}} - (k+1) \frac{1}{\hat{\lambda}^2} +$$

$$+ \frac{1}{\hat{\lambda}^4} \cdot \sum_{i=1}^{m+k} t^2_{(i)} \tag{22}$$

which may be rearranged as :

$$(n-m-k) \frac{t^2_{(m+k)}}{\hat{\lambda}^4} - \frac{m-1}{2\hat{\lambda}^2} \cdot t_{(m)}\ \Lambda(t_{(m)}) - \frac{k+1}{\hat{\lambda}^2} + \frac{1}{\hat{\lambda}^4} \cdot \sum_{i=m}^{m+k} t^2_{(i)} = 0 \tag{23}$$

where by $\Lambda(t_{(m)})$ we understand the hazard rate :

$$\Lambda(t) = \frac{f(t\ ;\lambda)}{1-F(t;\lambda)} \tag{24}$$

calculated in the point $t_{(m)}$.

We obtain finally :

$$\hat{\lambda}^2 = \frac{(n-m-k)\, t^2_{(m+k)} + \sum_{i=m}^{m+k} t^2_{(i)}}{k+1+\frac{1}{2}(m-1)\, t_{(m)} \cdot \Lambda(t_{(m)})} \tag{25}$$

It is easy to see that if $m=1$ and $m+k=n$, the classical m.l.e. for λ^2 is obtained. (For linear estimations of λ, see Iliescu-Vodă, 1973).

4. Testing statistical hypotheses. **We shall construct now a SPRT for testing the simple hypothesis :**

$$H_o : \lambda = \lambda_o \qquad \text{versus} \qquad H_1 : \lambda = \lambda_1 \; (\lambda_o < \lambda_1) \tag{26}$$

Up to a certain constant, this one can be interpreted as a hypothesis on the mean-life of the considered Rayleigh process.

Let $t_1, t_2, \ldots, t_n, \ldots$ be a sequential sample drawn from T and let $Prob\{H_1 \mid H_o\} = \alpha$ and $Prob\{H_o \mid H_1\} = \beta$ be the usual risks in sequential analysis.

Straightforward computations provide the log-likelihood ratio as :

$$ln \frac{\prod_{i=1}^{n} f(t_i;\lambda_1)}{\prod_{i=1}^{n} f(t_i;\lambda_o)} = ln\, r_n = 2n\, ln \frac{\lambda_o}{\lambda_1} + \left(\frac{1}{\lambda_o^2} - \frac{1}{\lambda_1^2}\right) \sum_{i=1}^{n} t_i^2 \tag{27}$$

Therefore, in accordance with Wald's rules (1960) we can present the decisions as below :

(1) if $\sum_{i=1}^{n} t_i^2 \le \frac{ln \frac{\beta}{1-\alpha}}{\frac{1}{\lambda_o^2} - \frac{1}{\lambda_1^2}} + n \frac{2\, ln \frac{\lambda_1}{\lambda_o}}{\frac{1}{\lambda_o^2} - \frac{1}{\lambda_1^2}}$, **then accept H_o ;**

(2) if $\sum_{i=1}^{n} t_i^2 \geq \dfrac{ln \dfrac{1-\beta}{\alpha}}{\dfrac{1}{\lambda^2} - \dfrac{1}{\lambda_1^2}} + n \cdot \dfrac{2\, ln \dfrac{\lambda_1}{\lambda_o}}{\dfrac{1}{\lambda_o^2} - \dfrac{1}{\lambda_1^2}}$, then accept H_1 ;

(3) if $\dfrac{ln \dfrac{\beta}{1-\alpha}}{\dfrac{1}{\lambda_o^2} - \dfrac{1}{\lambda_1^2}} + n \dfrac{2\, ln \dfrac{\lambda_1}{\lambda_o}}{\dfrac{1}{\lambda_o^2} - \dfrac{1}{\lambda_1^2}} < \sum_{i=1}^{n} t_i^2 < \dfrac{ln \dfrac{1-\beta}{\alpha}}{\dfrac{1}{\lambda_o^2} - \dfrac{1}{\lambda_1^2}} + n \dfrac{2\, ln \dfrac{\lambda_1}{\lambda_o}}{\dfrac{1}{\lambda_o^2} - \dfrac{1}{\lambda_1^2}}$,

then continue the experiment.

The power of the test is given by $\pi(\lambda) = 1 - L(\lambda)$ where $L(\lambda)$ is the OC-function, defined as below :

$$L(\lambda) \doteq \frac{A^h - 1}{A^h - B^h}, \quad A \doteq \frac{1-B}{\alpha}, \quad B \doteq \frac{B}{1-\alpha}. \tag{28}$$

Here , h is given by the Wald's fundamental lemma, namely :

$$E(e^{hz}) = 1 \qquad (h \neq 0), \tag{29}$$

where

$$z = ln\, f(t; \lambda_1) - ln\, f(t; \lambda_o). \tag{30}$$

In our specific application we have :

$$z = ln \frac{f(t;\lambda_1)}{f(t;\lambda_o)} = 2\, ln \frac{\lambda_o}{\lambda_1} + \left(\frac{1}{\lambda_o^2} - \frac{1}{\lambda_1^2} \right) t^2. \tag{31}$$

Hence :

$$E(e^{hz}) = C \int_o^{\infty} t \exp\left[-\left(\frac{h}{\lambda_1^2} - \frac{h}{\lambda_o^2} + \frac{1}{\lambda^2} \right) t^2 \right] dt , \tag{32}$$

where $C = \frac{2}{\lambda^2}\left(\frac{\lambda_o}{\lambda_1}\right)^2$

Imposing $b\lambda_1^{-2} - b\lambda_o^{-2} + \lambda^{-2} > 0$ and denoting $(b\lambda_1^{-2} - b\lambda_o^{-2} + \lambda^{-2})t^2 = u$

we obtain immediately :

$$E(e^{zb}) = \frac{1}{\lambda^2}\left(\frac{\lambda_o}{\lambda_1}\right)^{2b}\left(\frac{b}{\lambda_1^2} - \frac{b}{\lambda_o^2}\right)^{-1} = 1 . \qquad (33)$$

From (33) it follows that :

$$\lambda^2 = \frac{\left(\frac{\lambda_o}{\lambda_1}\right)^{2b} - 1}{b\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_o^2}\right)} , \qquad (34)$$

which together with (28) represents a parametric form for the OC-function .

Now, we compute the ASN for the given test. We have :

$$E_\lambda(n) \doteq \frac{L(\lambda)\ ln\ B + (1-L(\lambda))\ ln\ A}{E_\lambda(z)} \qquad (35)$$

We have therefore :

$$E_\lambda(z) = 2\ ln\ \frac{\lambda_o}{\lambda_1} + \left(\frac{1}{\lambda_o^2} - \frac{1}{\lambda_1^2}\right) \cdot \frac{2}{\lambda^2} \int_o^\infty t^3 exp\{-t^2/\lambda^2\}\, dt \qquad (36)$$

Taking into account the formula :

$$\int y^n\, exp\{-y^2\}\, dy = -\frac{1}{2} y^{n-1} e^{-y^2} + \frac{1}{2}(n-1)\int y^{n-2}\, exp\{-y^2\}\, dy \qquad (37)$$

we obtain for $n=3$ - needed for (36) - the following expression for $E_\lambda(z)$:

$$E_\lambda(z) = 2\ ln\ \frac{\lambda_o}{\lambda_1} + (\lambda_o^{-2} - \lambda_1^{-2})\ \lambda^2 \tag{38}$$

Therefore :

$$E_\lambda(n) \doteqdot \frac{h_2 + (h_1 - h_2)\ L(\lambda)}{q + \lambda^2}\ , \tag{39}$$

where we have introduced following notations :

$$h_1 = \frac{ln\ \dfrac{\beta}{1-\alpha}}{\lambda_o^{-2} - \lambda_1^{-2}}\ ,\quad h_1 = \frac{ln\ \dfrac{1-\beta}{\alpha}}{\lambda_o^{-2} - \lambda_1^{-2}}\quad \text{and}\quad q = \frac{2\ ln\ \dfrac{\lambda_1}{\lambda_o}}{\lambda_o^{-2} - \lambda_1^{-2}} \tag{40}$$

We observe that the uncertainty domain of the test can be written as :

$$h_1 + n\,q < \sum_{i=1}^{n} t_i^2 < h_2 + n\,q\ , \tag{41}$$

that is, bounds for $\sum_{i=1}^{n} t_i^2$ **are respectively acceptance and rejection straight-lines .**

4.1. Sequential comparison. **Consider now two independent Rayleigh variables** X **and** Y **, namely :**

$$X :\ f(x\ ;\theta) = \frac{2}{\theta^2}\ x\ exp\ (-x^2/\theta^2)\ ,\ x \geq 0\ ,\ \ \theta > 0\ , \tag{42}$$

$$Y :\ f(y\ ;\omega) = \frac{2}{\omega^2}\ y\ \ exp\ (-y^2/\omega^2)\ ,\ \ y \geq 0\ ,\ \omega > 0\ , \tag{43}$$

and suppose that we have to test :

$$H :\ \theta \leq \omega \qquad \text{against} \qquad H' :\ \theta > \omega\ .$$

Up to a certain constant it can be interpreted as a comparison between the mean - life times of two Rayleigh processes .

To solve this problem we shall apply Girshick's procedure ; the likelihood ratio is in this case (see for details Wald, 1960) :

$$R_n = \frac{\prod_{i=1}^{n} f_1(x_i ; y_i)}{\prod_{i=1}^{n} f_o(x_i ; y_i)} = exp\left[\left(\frac{1}{\theta_o^2} - \frac{1}{\omega_o^2} \right) \sum_{i=1}^{n} (x_i^2 - y_i^2) \right] \tag{44}$$

with $\theta_o < \omega_o$, or

$$ln\ R_n = \delta \sum_{i=1}^{n} (x_i^2 - y_i^2) \tag{45}$$

where :

$$\delta = -v(\theta_o ; \omega_o) = \theta_o^{-2} - \omega_o^{-2} \quad ; \; v(\theta ; \omega) = \omega^{-2} - \theta^{-2}$$

being the so called Girschick's function.

The test itself is as follows :

(a) if $\sum_{i=1}^{n} (x_i^2 - y_i^2) \leq \frac{1}{\delta}\ ln\ \frac{\beta}{1-\alpha}$ then accept H ;

(b) if $\sum_{i=1}^{n} (x_i^2 - y_i^2) \geq \frac{1}{\delta}\ ln\ \frac{1-\beta}{\alpha}$ then accept H' ;

(c) if $\frac{1}{\delta}\ ln\ \frac{\beta}{1-\alpha} < \sum_{i=1}^{n} (x_i^2 - y_i^2) < \frac{1}{\delta}\ ln\ \frac{1-\beta}{\alpha}$ then continue the experiment .

To compute OC - function and the ASN we need first the following .

LEMMA 2 : If X and Y are two independent Rayleigh variables, then Wald's constant $h(\theta ; \omega)$ is given by :

$$h(\theta ; \omega) = \frac{(\theta^2 - \omega^2)(\omega_o \theta_o)^2}{(\theta_o^2 - \omega_o^2)(\omega\, \theta)^2} \tag{46}$$

Proof : Using Obreja's reduction (1969) of Wald's equation

$$E\,[\,exp\,\{\,z\,h(\theta\,;\,\omega)\,\}\,] = 1 \tag{47}$$

We shall obtain :

$$E\,[\,exp\,\{\delta\,h(\theta\,;\omega)\,x^2\}]\,E\,[\,exp\,\{-\,\delta\,h(\theta\,;\omega)\,y^2\,\}] = 1 \tag{48}$$

which provides easily :

$$[\,1-\theta^2 h(\theta\,;\omega)\,\delta\,]^2 \cdot [\,1+\omega^2 h(\theta\,;\omega)\,\delta\,]^2 = 1 \tag{49}$$

from it follows immediately :

$$\delta\,h\,(\theta\,;\,\omega) = \frac{1}{\omega^2} - \frac{1}{\theta^2} \tag{50}$$

which gives the required relation. Lemma is proved .

Hence, the OC - function has the representation :

$$L\,(\theta\,;\,\omega) \doteq \frac{A^{\frac{(\theta^2-\omega^2)(\theta_o\omega_o)^2}{(\theta_o^2-\omega_o^2)(\theta\omega)^2}} - 1}{A^{\frac{(\theta^2-\omega^2)(\theta_o\omega_o)^2}{(\theta_o^2-\omega_o^2)(\theta\omega)^2}} - B^{\frac{(\theta^2-\omega^2)(\theta_o\omega_o)^2}{(\theta_o^2-\omega_o^2)(\theta\omega)^2}}} \tag{51}$$

To compute more rapidly the ASN we use Obreja - Vodă's form (1973) for the denominator of $E_{\theta\,;\omega}\,(n)$. Direct computations provide :

$$E_{\theta\,;\omega}\,(n) \doteq \frac{(\theta_o\omega_o)^2\,[\,L\,(\theta\,;\omega)\,ln\,B+(1-L\,(\theta\,;\omega)\,)\,ln\,A\,]}{(\omega_o^2-\theta_o^2)\,(\,\omega^2-\theta^2)} \tag{52}$$

The test now is completely constructed .

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