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ON RATIOS OF CERTAIN ANALYTIC FUNCTIONS

by

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ABSTRACT

Let $H(z) = f(z)/g(z)$ and $h(z) = f'(z)/g'(z)$ where both f and g are in one of the classes whose members are convex, starlike relative to the origin, or typically-real in the open unit disk E. Sharp bounds on $\alpha>0$ and $\beta>0$ such that $\Re e\{H(z)\}^{\alpha}>0$, $\Re e\{h(z)\}^{\beta}>0$, $z\in E$, are obtained with one exception. In the case where $h(z)$ is considered for f and g typically-real, z is restricted to the precise region in E of univalence of the class of typically-real functions.

1. ***Introduction*.** **Let A denote the analytic functions $f(z) = z + a_2 z^2 + \cdots$ in $E = \{z : |z| < 1\}$. Let K, S^*, and T respectively denote the subclasses of A whose functions are convex, starlike with respect to the origin, and typically-real in E ([5], [6]).**

A number of authors (e.g., [1], [2]) have treated problems of the following type: Fund the radius of starlikeness and the radius of convexity of the class of functions $f\in A$ such that $\Re e\{f'(z)/g'(z)\}>0$, $z\in E$, for each g contained in a certain subclass of A. In this paper, we assume f and g are in one of the

classes K, S^*, or T and determine the largest positive value of α such that $\Re e\{h(z)\}^{\alpha} > 0$, $z \in E$, where $h(z) = f'(z)/g'(z)$, $\{h(0)\}^{\alpha} = 1$. Also the largest positive value of β is determined such that $\Re e\{H(z)\}^{\beta} > 0$, $z \in E$, where f and g are in one of the cited classes and $H(z) = f(z)/g(z)$, $\{H(0)\}^{\alpha} = 1$. In particular, we obtain the following sharp results.

THEOREM 1. If $f, g \in K$, *then* $\Re e\{H(z)\}^{1/2} > 0$ *and* $\Re e\{h(z)\}^{1/4} > 0$ *for* $z \in E$.

THEOREM 2. If $f, g \in S^*$, *then* $\Re e\{H(z)\}^{1/4} > 0$ *and* $\Re e\{h(z)\}^{1/6} > 0$ *for* $z \in E$.

THEOREM 3. If $f, g \in T$, *then* $\Re e\{H(z)\}^{1/2} > 0$ *for* $z \in E$ *and* $\Re e\{h(z)\}^{1/4} > 0$ *for* z *in the exact region* D *of univalence of the class* T.

Each $f \in T$ is known [4] to be univalent in the region $D \subset E$ defined by

$$(1) \qquad \{z : \Re e\{z/(1-z)^2\} > -1/4\}.$$

Moreover, if $z_0 \in E$, $z_0 \notin D$, it can be shown (see [4]) that there exists a function $f \in T$ for which $f'(z_0) = 0$. This implies that $h(z) = f'(z)/g'(z)$, where $f, g \in T$, is not always analytic outside the region D. It is, therefore, necessary to restrict attention to the region D for the second part of Theorem 3.

2. *Proof of Theorem 1.* For $f, g \in K$, $z \in E$, we have [3] that $f'(z)$ and $g'(z)$ are in the image of E by the mapping $w = 1/(1-z)^2$. Hence, we have

$$|\arg h(z)| \leq |\arg f'(z)| + |\arg g'(z)| \leq 4 \arcsin |z|.$$

This implies $\Re e\{h(z)\}^{1/4} > 0$ for $z \in E$. For sharpness, let $f(z) = z/(1-z)$

and $g(z) = z/(1+e^{i\phi}z)$, $-\pi < \phi \leq \pi$. Then $h(z) = (1+e^{i\phi}z)^2/(1-z)^2$. The function $\{h(z)\}^{\frac{1}{2}}$ maps E onto a half-plane bounded by a line through the origin with angle of inclination $(\pi+\phi)/2$. Hence, for each $\alpha > 1/4$ there is a choice of ϕ, $-\pi < \phi < \pi$, such that $\Re\{h(z)\}^{\alpha} < 0$ for some $z \in E$.

For the second result in the theorem we use the fact [8] that $|\arg f(z)/z| < \arcsin |z|$, $z \in E$, when $f \in K$. Thus, if f and g are in K, then $|\arg H(z)| < 2 \arcsin |z|$ so $\Re\{H(z)\}^{1/2} > 0$ for $z \in E$. Sharpness is verified using the same functions as in the previous part of the theorem.

3. *Proof of Theorem 2.* The function $f \in S^*$ if and only if $F(z) = \int_0^z (f(t)/t)dt \in K$. It follows from Theorem 1 that $\Re\{H(z)\}^{1/4} > 0$, $z \in E$ and that this result is sharp.

For the other part of the theorem, we have

$$
\begin{aligned}
|\arg h(z)| &= |\arg f'(z) - \arg g'(z)| \\
\text{(2)} \qquad &\leq \left|\arg \frac{zf'(z)}{f(z)}\right| + \left|\arg \frac{zg'(z)}{g(z)}\right| + \left|\arg \frac{g(z)}{f(z)}\right|.
\end{aligned}
$$

Now by the first part of the theorem $|\arg\{g(z)/f(z)\}| < 2\pi$, $z \in E$. Since $|\arg\{zf'(z)/f(z)\}| < \pi/2$, $z \in E$, whenever $f \in S^*$, we conclude from (2) that

$$|\arg h(z)| \leq \frac{\pi}{2} + \frac{\pi}{2} + 2\pi = 3\pi.$$

Thus, $\Re\{h(z)\}^{1/6} > 0$, $z \in E$. To establish the sharpness, let $f(z) = z/(1-z)^2$, $g(z) = z/(1+e^{i\phi}z)^2$, $-\pi < \phi \leq \pi$. Then $h(z) = AB^3$ where $A = (1+z)/(1-e^{i\phi}z)$ and $B = (1+e^{i\phi}z)/(1-z)$. For $z = e^{-i\phi/2}$ and for $\alpha > 1/6$, there is a ϕ, $\pi/2 < \phi < \pi$, such that

$$\alpha \arg (AB^3) = \alpha \frac{\pi - \phi}{2} + 3\alpha \left(\frac{\pi + \phi}{2} \right) = (2\pi + \phi)\alpha > \pi/2 .$$

Indeed, select any ϕ in the interval $-\pi/2\alpha - 2\pi < \phi < \pi$. It follows that $\Re e \{ h(z) \}^{\alpha} > 0$ for some $z \in E$ when $\alpha > 1/6$ and $h(z) = AB^3$.

4. *The typically-real case*. The first part of Theorem 3 is proved using the result of Rogosinki [6] that states $f \in T$ if and only if there is an analytic function p in E such that $p(z)$ is real for real values of $z \in E$, $p(0) = 1$, $\Re e\, p(z) > 0$ for $z \in E$, and $f(z) = zp(z)/(1-z^2)$. It follows that the function $H(z) = f(z)/g(z)$, where $f, g \in T$, can be expressed as the ratio of two normalized functions, each with positive real part in E. Hence, $\Re e \{ H(z) \}^{1/2} > 0$ for $z \in E$. The sharpness is established by setting $f(z) = z/(1-z)^2$ and $g(z) = z/(1+z)^2$.

For the second part of Theorem 3, we use a result of Merkes [4] that states $f \in T$ if and only if there exists a nondecreasing function $\gamma(t)$, $\gamma(1) - \gamma(0) = 1$, such that

$$(3) \qquad F(\zeta) = \int_0^1 \frac{\zeta\, d\,\gamma(t)}{1 + \zeta t}$$

where $\zeta = 4z/(1-z)^2$ and F is defined by $4f(z) = F(\zeta)$. By an elementary argument it is easily proved that $|\arg F'(\zeta)| < \pi$ for $\Re e\, \zeta > -1$, indeed, for $\Re e\, \zeta > -1$, we have

$$-\pi < \arg (1 + t\zeta)^{-2} \leq 0 \qquad (\Im m\, \zeta > 0),$$

$$0 \leq \arg (1 + t\zeta)^{-2} < \pi \qquad (\Im m\, \zeta < 0)$$

Thus, for $\Re e\, \zeta > -1$, $\Im m\, \zeta > 0$, we have by (3) $-\pi \leq \arg F'(\zeta) = \int_0^1 \arg (1+\zeta t)^{-2}\, d\gamma(t) \leq 0$. Similarly, for $\Im m\, \zeta < 0$, $\Re e\, \zeta > -1$ we have $0 \leq \arg F'(\zeta) \leq \pi$.

Now, for $f, g \in T$, we have

$$\frac{f'(z)}{g'(z)} = \frac{F'(\zeta)}{G'(\zeta)}, \quad z \in E,$$

where $\zeta = 4z/(1-z)^2$ and $4f(z) = F(\zeta)$, $4g(z) = G(\zeta)$. If $\Re e\, \zeta > -1$, i.e., if $z \in D$, where D is defined by (1), then

$$\frac{1}{4}\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{1}{4}\left(\left|\arg F'(\zeta)\right| + \left|\arg G'(\zeta)\right|\right) \leq \frac{\pi}{2}$$

Thus, $\Re e\{h(z)\}^{1/4} > 0$ for $z \in D$. To establish the sharpness, let $f(z) = z/(1-z)^2$ and $g(z) = z/(1+z)^2$. For these functions, $h(z) = (1+z)^4/(1-z)^4 = (1+\zeta)^2$, and, hence, $\Re e\{h(z)\}^{\alpha} > 0$ for $z \in D$ if and only if $\alpha \leq 1/4$ when $\alpha > 0$.

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