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ON WARING'S PROBLEM IN ALGEBRAIC NUMBER FIELDS

by

Yosikazu EDA

Dedicated to Professor Tikao Tatuzawa on his 60th birthday

§ 1. *Introduction.* Let K be an algebraic number field of order n over the rational number field Q and k be a positive rational integer. Let $\mathfrak{o}$ be the integral domain consisting of all integers in K (the unit ideal of K) and J_k be the ring generated by k-th powers of all integers in K. On account of the identity

$$k!\,\gamma = \sum_{\ell=0}^{k-1} (-1)^{k-1-\ell} \binom{k-1}{\ell} \{ (\gamma+\ell)^k - \ell^k \}$$

for $\gamma \in \mathfrak{o}$, the ring J_k is an order. We use a letter c (and similarly c', c'', $\ldots$, c_1, c_2, $\ldots$) to denote a positive constant depending only on K, k and s. It is not necessarily the same one each time it occurs. The constant c may well depend on another parameter $*$. In this case we write $c(*)$ to explain the meaning.

Applying the Vinogradov [13], [14], [15], Siegel [7], [8] method and using the results of Tatuzawa [10], [11], [12], Körner [4], Stemmler [9] and Mitsui [5], [6], we shall prove the following:

MAIN THEOREM: Let $k \geqq c$ (which is determined explicitly) and let $\nu \epsilon J_k$ be a totally positive integer with sufficiently large norm $N(\nu)$. If

$$s > 2nk\log k + 6nk\log\log k + 2(2n\log n + n\log\log n + 2\log n + 14)k + 1,$$

then the equation

$$\nu = \lambda_1^k + \lambda_2^k + \cdots + \lambda_s^k$$

is always solvable in totally non-negative integers λ_r $(1 \leqq r \leqq s)$ subject to the conditions

$$N(\lambda_r)^k \leqq c' N(\nu) \qquad (1 \leqq r \leqq s).$$

We introduce the notation $G_K(k)$ for the least value of s such that every totally positive integer ν, where $N(\nu)$ is sufficiently large, is representable in the form of the theorem.

In the rational number field Q, $G(k)$ denote the least value of s with the property that every sufficiently large rational integer ν is representable in the sum of s k-th powers where $\lambda_1, \ldots, \lambda_s$ are non-negative rational integers.

Siegel seceeded in dealing with the Waring problem in an algebraic number field. He extended the circle method of Hardy-Littlewood theory to the case of an algebraic number field.

In 1958 Tatuzawa [10] proved that

$$G_K(k) < 8nk(n+k)$$

and in 1971 [11], [12], he obtained the result between $G_K(k)$ and n i.e.

$$G_K(k) \leqq 4nk + 2nG(k) + 1 .$$

In 1961 Korner [4] proved that

$$G_K(k) < nk\left(3\log k + 3\log\left(\frac{n^2+1}{3}\right) + 1\right).$$

In 1959 it has shown by I. M. Vinogradov [14] himself that for $k \geqq 170{,}000$

$$G(k) < 2k\log k + 4k\log\log k + 2k\log\log\log k + 13k .$$

Our aim is to extend this remarkable and very beatiful result to an algebraic number field. Our results were reported at the United States-Japan Seminar on Modern Methods in Number Theory at Tokyo in 1971 [2].

§ *2. Singular Series and Generalized Farey Dissection.*

Let $K^{(\ell)}$ $(1 \leqq \ell \leqq n_1)$ be n_1 real conjugate fields, and let $K^{(m)}, K^{(m+n_2)}$ $(n_1+1 \leqq m \leqq n_1+n_2)$ be n_2 pairs of complex conjugate fields, so that $n = n_1 + 2n_2$. We denote by $\gamma^{(q)}$ $(1 \leqq q \leqq n)$ the conjugates of $\gamma \in K$ and define

$$S(\gamma) = trace(\gamma) = \sum_{q=1}^{n} \gamma^{(q)} .$$

Let γ_r $(1 \leqq r \leqq n)$ be numbers of K and x_r $(1 \leqq r \leqq n)$ be real numbers. We set $\xi = \sum_{r=1}^{n} \gamma_r x_r$ and define

$$\xi^{(q)} = \sum_{r=1}^{n} \gamma_r^{(q)} x_r \quad \text{and} \quad trace(\xi) = \sum_{q=1}^{n} \xi^{(q)} .$$

For brevity we write $E(\xi) = exp\{2\pi i \; trace(\xi)\}$ and use the abbreviations

$$||\gamma|| = \underset{1 \leqq q \leqq n}{Max} |\gamma^{(q)}| \quad \text{and} \quad ||\xi - \gamma|| = \underset{1 \leqq q \leqq n}{Max} |\xi^{(q)} - \gamma^{(q)}|$$

A number γ of K is called totally positive or totally non-negative according to $\gamma^{(\ell)} > 0$ or $\gamma^{(\ell)} \geqq 0$ $(1 \leqq \ell \leqq n_1)$ respectively. If F and G are functions of certain variables and G is positive, then the notation $F \ll G$ or $F = O(G)$ means that there exists a positive constant c (O-constant) such that $|F| \leqq cG$ in the domain designated.

Let $\mathfrak{d}$ be the ramification ideal of K, and D be the discriminant of K, so that $N(\mathfrak{d}) = D$. For any number γ in K, we can determine uniquely integral ideals $\mathfrak{a}$, $\mathfrak{b}$ such that

$$\gamma\,\mathfrak{d} = \mathfrak{b}/\mathfrak{a} \qquad , \qquad (\mathfrak{a}, \mathfrak{b}) = \mathfrak{o} .$$

We write then $\gamma \to \mathfrak{a}$ for convenience and we call $\mathfrak{a}$ denominator of γ. We can choose $[\omega_1, \ldots, \omega_n]$ as the integral basis of K and $[\rho_1, \ldots, \rho_n]$ as a basis of $\mathfrak{d}^{-1}$ satisfying

$$trace(\rho_r \omega_s) = \begin{cases} 1 & (r = s) \\ 0 & (r \neq s). \end{cases}$$

Since, by definition $\mathfrak{d}^{-1} = \{\rho \,;\, E(\rho\alpha) = 1 \text{ for every } \alpha \in \mathfrak{o}\}$, we obtain $E(\lambda^k\gamma) = E(\mu^k\gamma)$ provided $\gamma \to \mathfrak{a}$, $\lambda, \mu \in \mathfrak{o}$ and $\lambda \equiv \mu \pmod{\mathfrak{a}}$. Let

$$S(\gamma) = \sum_{\lambda} E(\lambda^k\gamma) ,$$

the summation being over a complete residue system mod $\mathfrak{a}$. In view of the above, the sum $S(\gamma)$ is independent of the choice of a system.

Let $\nu \in \mathfrak{o}$ and let $\mathfrak{a}$ be an integral ideal in K. We constitute the sum

$$H(\mathfrak{a}) = \sum_{\gamma} N(\mathfrak{a})^{-s}\, S(\gamma)^s \quad E(-\nu\gamma) ,$$

where γ runs over a reduced residue system of $(\mathfrak{a}\,\mathfrak{d})^{-1}$ mod $\mathfrak{d}^{-1}$. $H(\mathfrak{a})$ is independent of the choice of a system. The series

$$\mathfrak{S}(\nu) = \sum_{a} H(a)$$

is called a singular series which is absolutely convergent for $s \geqq 4kn$.

The following theorem is one of the fundamental theorem to Waring problem.

THEOREM 1. Assume that $k \geqq 3$, $s \geqq s_o = 4nk$ *and* ν *is an integer of* J_k *Then the singular series* $\mathfrak{S}(\nu)$ *is absolutely convergent and can be expressed as*

$$\mathfrak{S}(\nu) = \prod_{\mathfrak{p}} \chi_{\mathfrak{p}}(\nu)$$

where

$$\chi_{\mathfrak{p}}(\nu) = \sum_{\ell=o}^{\infty} H(\mathfrak{p}^{\ell}).$$

Furthermore there exists positive constants c', c'' *satisfying*

$$c' > \mathfrak{S}(\nu) > c''.$$

Tatuzawa [10] [12] proved this result for $s \geqq s_o = [8nk(\log k-1)]$. Stemmler [9] improved on the estimation of s_o as mentioned above.

Let X be the n-dimensional Euclidean space and U be the unit cube, namely

$$U = \{(x_1, x_2, \ldots, x_n) \;;\; 0 \leqq x_1 \leqq 1, \ldots, 0 \leqq x_n \leqq 1\}.$$

We denote by Γ the set of

$$\gamma = \rho_1 x_1 + \cdots + \rho_n x_n$$

fulfilling the conditions

$$(x_1, \ldots, x_n) \in U,\ x_r\ (1 \leqq r \leqq n) \text{ rational numbers},\quad N(a) \leqq t^n,$$

where $\gamma \to a$. We define the basic domain $B_\gamma = B_\gamma(t, h)$, for every $\gamma \in \Gamma$, subject to $\gamma \to a$, by

$$x = (x_1, \dots, x_n) ; \quad x \in U , \quad \xi = \rho_1 x_1 + \cdots + \rho_n x_n$$

$$\{ \prod_{q=1}^{n} Max (h \mid \xi^{(q)} - \gamma_o^{(q)} \mid , \ t^{-1}) \leqq 1/ N(\mathfrak{a}) ,$$

for any $\gamma_o \equiv \gamma \mod \mathfrak{d}^{-1}$, whose contribution to ξ does not vanish $\}$.

$S = S(t, h) = U - \sum_{\gamma \in \Gamma} B_\gamma$ is termed the supplementary domain. We write in these cases as $\xi \in B_\gamma$ or $\xi \in S$. This division of U into $S = S(t, h)$ and $B_\gamma = B_\gamma(t, h)$ depends on the pair (t, h). We shall call this division the generalized Farey dissection of U with respect to (t, h).

Suppose that

$$\xi = \rho_1 x_1 + \cdots + \rho_n x_n , \qquad (x_1, \dots, x_n) \in X$$

$$\eta = \omega_1 y_1 + \cdots + \omega_n y_n , \qquad (y_1, \dots, y_n) \in Y ,$$

then we use abbreviations

$$dx \quad \text{or} \quad d\xi = dx_1 \dots dx_n ,$$

$$dy \quad \text{or} \quad d\eta = dy_1 \dots dy_n .$$

Let T' be a sufficiently large number. We denote by $k \geqq c$ a fixed natural number and write

$$P = [T'^{1/k}] , \ X_o = [P] , \quad R = [X_o^{1 - 1/(2\sqrt{k})}] ,$$

$$t_1 = X_o^{1-f} , \ h_1 = X_o^{k-1+f} \qquad (0 < f \leqq \tfrac{1}{4})$$

$$t_o = R^{h} , h_o = R^{k} t_o^{n-1} \log t_o , \ \text{where} \quad h = n/(1 + n^2) .$$

We use the generalized Farey dissection of U with respect to (t, h), where $(t, h) = (t_1, h_1)$ or (t_o, h_o).

The following fundamental lemmas were proved by Siegel [8].

LEMMA 1. If $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$ *and* T' *is sufficiently large, then* $B_{\gamma_1} \cap B_{\gamma_2} = \phi$.

LEMMA 2. If $(x_1, \ldots, x_n) \in S$, *then there exist an integer* $\alpha \in \mathfrak{o}$ *and* $\beta \in \mathfrak{d}^{-1}$ *satisfying the following conditions :*

(i) $\| \alpha \xi - \beta \| < h^{-1}$,

(ii) $t < \| \alpha \| \leqq h$,

(iii) $Max\,(h \mid \alpha^{(q)} \xi^{(q)} - \beta^{(q)} \mid , \mid \alpha^{(q)} \mid) \geqq D^{-\frac{1}{2}}$ $(1 \leqq q \leqq n)$

(iv) $N((\alpha, \beta\mathfrak{d})) \leqq D^{\frac{1}{2}}$.

We write

$$L(\xi) = \sum_{\lambda \in \Omega(T)} E(\lambda^k \xi), \qquad \xi = \rho_1 x_1 + \cdots + \rho_n x_n ,$$

where $\lambda = \omega_1 z_1 + \cdots + \omega_n z_n \in \mathfrak{o}$ runs over all integers restricted as

$$o \leqq \lambda^{(\ell)} \leqq T \qquad (1 \leqq \ell \leqq n_1)$$

$$| \lambda^{(m)} | \leqq T \qquad (n_1 + 1 \leqq m \leqq n_1 + n_2)$$

and in this case we use the notation as $\lambda \in \Omega(T)$.

Then the number of solutions of

$$\lambda_1^k + \lambda_2^k + \cdots + \lambda_s^k = \nu , \quad \lambda_j , \nu \in \mathfrak{o}$$

is expresseible by the integral

$$I(\nu) = \int_U \cdots \int L(\xi)^s E(-\nu\xi)\, dx .$$

From Lemma 1 of Siegel we divide the integral as follows :

$$I(\nu) = \sum_{\gamma \in \Gamma} \int_{B_\gamma} \cdots \int + \int_S \cdots \int .$$

We denote by $P(T)$ the set of $(x_1, \ldots, x_n)$ or η satisfying

$$0 \leqq \eta^{(\ell)} \leqq T \qquad (1 \leqq \ell \leqq n_1)$$

$$|\eta^{(m)}| \leqq T \qquad (n_1 + 1 \leqq m \leqq n_1 + n_2)$$

and in this case we use the same notation as $\eta \in P(T)$.

THEOREM 2. We write

$$J(\mu) = \int_X \cdots \int E(-\mu\xi)\left(\int_P \cdots \int E(\xi\eta^k)\,dy\right)^s dx$$

where $P = P(1)$ *and* $\mu \neq 0$, $\mu \in P(1)$.

By means of Fourier transformation, we have

$$J(\mu) = |D|^{(1-s)/2} \prod_{\ell=1}^{r_1} F(\mu^{(\ell)}) \prod_{m=n_1+1}^{n_1+n_2} H(\mu^{(m)}) \qquad (s > k)$$

where

$$F(\mu^{(\ell)}) = \int_{D_\ell} \cdots \int \prod_{j=1}^{s} \left(k^{-1} u_j^{\frac{1}{k}-1}\right) du_1 \ldots du_{s-1}, \quad u_s = \mu^{(\ell)} - (u_1 + \cdots + u_{s-1}),$$

D_ℓ *being the closed region determined by*

$$0 \leqq u_j \leqq 1 \qquad (1 \leqq j \leqq s).$$

We have

$$F(\mu^{(\ell)}) = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} (\mu^{(\ell)})^{s/k-1}$$

Furthermore

$$H(\mu^{(m)}) = k^{-s} \int_{D_m} \cdots \int \prod_{j=1}^{s} (k^{-1} u_j^{1/k-1})\, du_1 \ldots du_{s-1}\, d\varphi_1 \ldots d\varphi_{s-1}$$

$$u_s = |\mu^{(m)} - (u_1^{\frac{1}{2}} e^{i\varphi_1} + \cdots + u_{s-1}^{\frac{1}{2}} e^{i\varphi_{s-1}})|^2$$

D_m *being the closed region determined by*

$$0 \leqq u_j \leqq 1 \quad (1 \leqq j \leqq s), \; -\pi \leqq \varphi_j \leqq \pi \qquad (1 \leqq j \leqq s-1).$$

Proof: See Tatuzawa [12], Siegel [8].

For the basic domain we have the following

THEOREM 3. (Tatuzawa [12], Siegel [8]). *If* $s \geqq 4nk$, *then*

$$\sum_{\gamma \epsilon \Gamma} \int_B \cdots \int L(\xi)^s \, E(-\nu\xi) \, dx = \mathfrak{S}(\nu) J(\mu) T^{n(s-k)} + O(T^{n(s-k)-\frac{1}{4}}),$$

provided $\nu \epsilon \Omega(T^k)$ *and* $\mu = \nu T^{-k}$.

This is one of the fundamental theorems.

§3. *Estimations of some trigonometrical sums.*

Let T' be a sufficiently large integer. We denote by $k \geqq c$ a fixed integer and write $P = [T'^{1/k}]$, $X_o = [P]$. Furthermore we write

$$(1) \qquad Y_o = [\sqrt{X_o}], \; X_h = [X_o^{\delta^h}], \; Y_h = [\sqrt{X_h}] \quad (1 \leqq h \leqq s_1),$$

$$\| \sigma_h \| < Y_h ,$$

where

$$(2) \qquad s_1 > 2kn, \; \delta = \frac{1 - s_o/2}{k - \frac{1}{2}}, \qquad s_o = [\tfrac{1}{2} \log k].$$

Let

$$(3) \qquad r = 2r_o, \quad r_o \geqq \tfrac{1}{2}(1 + 1/(n-1))\,k\,(k+1) + \tau k - 1,$$

where $\tau \geqq 0$ is a natural number.

We call an integer $\omega \in \mathfrak{o}$ a prime number, if the principal ideal (ω) is a prime ideal .

Let $P = P(R, c)$ be the set of total positive prime numbers such that

$$(4) \quad \begin{aligned} &\tfrac{1}{2} cR \leqq |\omega^{(\ell)}| \leqq cR \qquad (1 \leqq \ell \leqq n) \\ &|\arg \omega^{(m)}| \leqq \tfrac{1}{4} \pi / R \qquad (n_1 + 1 \leqq m \leqq n_1 + n_2) . \end{aligned}$$

We denote by $P^*(R, c)$ the number of elements of the set $P(R, c)$. By Mitsui [5] we have

LEMMA 3. $$1 \ll P^*(R, c) R^{-n} \log R \ll 1$$

Let $Q(s_1, X_o)$ be the set of $\mu \in \mathfrak{o}$ such that

$$\mu = (X_o + \sigma_1)^k + \cdots + (X_{s_1} + \sigma_{s_1})^k$$

$$0 < |\sigma_i| < Y_i \qquad (1 \leqq i \leqq s_1).$$

We denote by $Q^*(s_1, X_o)$ the number of elements of the set $Q(s_1, X_o)$, and we put

$$Q' = Q^*(s_1, X_o) P^*(R, c).$$

The purpose of this section is to estimate the following trigonometrical sum $R(\xi)$ in the supplementary domain $S(t, h) \ni \xi$.

THEOREM 5 . We shall define a trigonometrical sum

$$R(\xi) = \sum_{\mu \in Q(s_1, X_o)} \sum_{\omega \in P(R,c)} E(\mu \omega^k \xi) ,$$

where $\xi \in S(t, h)$ and the right-hand side is taken over all integers μ and prime numbers such that

$$\mu \in Q(s_1, X_o) \quad , \quad \omega \in P(R, c).$$

Then we have

$$R(\xi) \ll Q' X_o^{\rho} \quad ,$$

where

$$\rho = \frac{1}{2r_o}\left(\left(k - \tfrac{1}{2}\right)\delta^{s_1} + \tfrac{1}{2}(s_o + 1)(1 - 1/s_o)^{\tau}(1 - \delta^{s_1})(k-2) - \tfrac{1}{2}(b/n), n < 0.\right.$$

LEMMA 4. Let $k \geqq c$ and λ_i, μ_i $(1 \leqq i \leqq s)$ be totally non-negative integers in K such that $\lambda_i, \mu_i \in \Omega(Q)$ and $N_s(Q)$ be the number of the solutions of which satisfy the system of Diophantine equations

$$(5) \quad \begin{cases} \lambda_1 + \cdots + \lambda_{s-1} + \lambda_s - \mu_1 - \cdots - \mu_s = 0 \\ \lambda_1^2 + \cdots + \lambda_s^2 - \mu_1^2 - \cdots - \mu_s^2 = 0 \\ \cdots\cdots\cdots\cdots \\ \lambda_1^k + \cdots + \lambda_s^k - \mu_1^k - \cdots - \mu_s^k = 0 \end{cases}$$

Let θ be a positive number and R be a sufficiently large real number such that

$$0 < \theta \leq 1 \quad , \quad R > Max\left(k, \left(2^n \theta^{-1}\right)^{1/k}\right),$$

and

$$(6) \quad (2R)^{nk\left(1 + \frac{1}{k-1}\right)^{\tau - 1}} < Q^n,$$

$$s \geqq \frac{n}{n-1} k(k+1) + \tau k - 1$$

where τ is a natural number.

Then we have

$$N_s(Q) \leqq c(k, \tau, s, n, K)\, Q^{2sn - \frac{1}{2}k(k+1)n + \frac{1}{2}k(k+1)(1 - 1/k)^{\tau} n}.$$

Proof. (Eda [1]). Let $\mathfrak{p}_\nu$ be a prime ideal of K. Then we obtain an integer π_ν with the conditions that $\pi_\nu \in \mathfrak{p}_\nu$, $\pi_\nu \notin \mathfrak{p}_\nu^2$ (in this case we write briefly $\mathfrak{p}_\nu \| \pi_\nu$) and

$$(7) \quad \begin{cases} c_1 \sqrt[n]{N\mathfrak{p}} \leqq |\pi_\nu^{(j)}| \leqq c_2 \sqrt[n]{N\mathfrak{p}} \quad (1 \leqq j \leqq n) \quad \text{and} \\ N\mathfrak{p}_\nu > k^n \qquad (1 \leqq \nu \leqq \tau) \end{cases}$$

Let P_ν $(1 \leqq \nu \leqq \tau)$ be positive real numbers such that

$$(8) \quad \begin{cases} c P_\nu^{1/k} > R \quad , \quad N\mathfrak{p}_\nu \cdot P_\nu > P_{\nu-1} \\ Q^{n(1-1/k)^\nu} \leqq P_\nu \leqq c_1(n,\nu)\, Q^{n(1-1/k)^\nu} \\ P_o = Q^n \quad (1 \leqq \nu \leqq \tau) \end{cases}$$

We put in (5)

$$\lambda_i = \varphi_i + \xi_i \pi_1 \quad , \quad \mu_i = \psi_i + \eta_i \pi_1 \, ,$$

$$\varphi_i \, , \psi_i \in \Omega(c_1 \pi_1) \qquad (1 \leqq i \leqq s) \, ,$$

where φ_i and ψ_i run over separately through a reduced system mod $\mathfrak{p}_\nu$ and are decided uniquely by λ_i and μ_i, respectively. Thus we obtain

$$(9) \quad (\varphi_1 + \xi_1 \pi_1)^j + \cdots + (\varphi_s + \xi_s \pi_1)^j = (\psi_1 + \eta_1 \pi_1)^j + \cdots + (\psi_s + \eta_s \pi_1)^j \qquad (1 \leqq j \leqq k)$$

$$\varphi_i + \xi_1 \pi_i \, , \psi_i + \eta_i \pi_1 \in \Omega(Q) \qquad (1 \leqq i \leqq s) \, .$$

We shall put the system

$$\{\varphi_1, \ldots, \varphi_s\} \quad , \quad \{\psi_1, \ldots, \psi_s\}$$

in the first class if we can find k distinct φ_i, ψ_i. All the remaining systems

$\{\varphi_1, \dots, \varphi_s\}, \{\psi_1, \dots, \psi_s\}$ will be placed in the second class.

Let $N_s^{(1)}(P)$ be the number of solutions of the system of equations (9) for $\{\varphi_1, \dots, \varphi_s\}$ and $\{\psi_1, \dots, \psi_s\}$ belong to the first class ,and let $N_s^{(2)}(P)$ be the number of solutions if (9) for either $\{\varphi_1, \dots, \varphi_s\}$ or $\{\psi_1, \dots, \psi_s\}$ belonging to the second class. By (8) we have

$$(10) \qquad N_s(P) \leqq N_s(N\mathfrak{p}_i P_1) = N_s^{(1)}(N\mathfrak{p}_1 P_1) + N_s^{(2)}(N\mathfrak{p}_1 P_1).$$

Furthermore we have

$$N_s^{(1)} \leqq \binom{s}{k}^2 k!\, c^{k^2 n} P_1^{2k} N\mathfrak{p}_1^{2s-\frac{1}{2}k(k+1)} N_{s-k}(P_1)$$

and

$$N_s^{(2)} \leqq 2k^s P_1^{2k} N\mathfrak{p}_1^{s-k-1} N_{s-k}(P_1) \, .$$

From the assumption (6) we obtain

$$\binom{s}{k}^2 k! > 2k^{2k} \qquad (k \geqq 2)$$

and

$$k^{s-2k} < N\mathfrak{p}_1^{s-\frac{1}{2}k(k+1)-k+1}$$

and from these results we have

$$(11) \qquad 2k^s N\mathfrak{p}_1^{s+k-1} < \binom{s}{k}^2 k!\, c^{k^2 n} N\mathfrak{p}_1^{2s-\frac{1}{2}k(k+1)} \qquad (c > 1)$$

Thus from (10) and (11) we obtain

$$N_s(P_o) \leqq 2c^{k^2 n}\binom{s}{k}^2 k!\, P_1^{2k} N\mathfrak{p}_1^{2s-\frac{1}{2}k(k+1)} N_{s-k}(P_1) \, .$$

and therefore we write this inequality as follows :

$$N_s(P_o) = D_1 P^{2k} N\mathfrak{p}_1^{2s-\frac{1}{2}k(k+1)} N_{s-k}(P_1) \, ,$$

$$D_1 = c^{k^2 n} \binom{s}{k}^2 k! \qquad (c > 2)$$

Applying this inequality τ times, we have

$$N_s(P) \leqq D_1 \dots D_\tau (P_1 \dots P_\tau)^{2k} (N\mathfrak{p}_1 \dots N\mathfrak{p}_\tau)^{2s - \frac{1}{2}k(k+1)}$$

$$(N\mathfrak{p}_1 \dots N\mathfrak{p}_\tau)^{\tau - 1 - 2k} P_\tau^{2(s - \tau k)}$$

From the definition D_i and the conditions (7) and (8) we obtain our lemma.

We denote by $M(T) = M(T, u, u', u'')$ the set of vectors $(z_1, \dots, z_n)$ or integers $\lambda = \omega_1 z_1 + \dots + \omega_n z_n$ satisfying the following conditions

$$\begin{cases} u \leqq \lambda^{(\ell)} < u' + T & \\ u' \leqq Re\ \lambda^{(m)} < u' + T & (1 \leqq \ell \leqq n_1) \\ u'' \leqq Im\ \lambda^{(m)} < u'' + T & (n_1 + 1 \leqq m \leqq n_1 + n_2). \end{cases}$$

LEMMA 5. We put

$$B_r = X_o^{k-1} z_1 + \dots + X_o^{k-r} \qquad (1 \leqq r \leqq k) \tag{12}$$

where

$$\| z_i \| \leqq c_i Y_o^i \qquad (1 \leqq i \leqq r) \tag{13}$$

and c_i *does not depend on* Y_o.

We denote by $n(B_r) \equiv n(z, B_r, X_o, Y_o)$ *the number of the integral solutions* $z = (z_1, \dots, z_r)$ *of (12) and (13) for a fixed* B_r.

Then we have

$$n(B_r) \ll Y_o^{(\frac{1}{2}r(r+1) - 1)n} X_o^{-rn + n}$$

Proof. If $r = 1$, then $B_1 = X_o^{k-1} z_1$. The number of z_1 is trivially estimated : $\ll 1$. We use induction and suppose that our result is true for $1 \leqq r \leqq \ell - 1$. We then deduce from this assumption our result for $r = \ell$.

If we have two solutions $(z_1, \ldots, z_\ell)$ and $(z'_1, \ldots, z'_\ell)$ satisfying the relations (12) and (13), then

$$B_\ell = X_o^{k-1} z_1 + \cdots + X_o^{k-\ell} z_\ell = B_{\ell-1} + X_o^{k-\ell} z_\ell = X_o^{k-1} z'_1 + \cdots + X_o^{k-\ell} z_\ell = B'_{\ell-1} + X_o^{k-\ell} z'_\ell .$$

From this we obtain

$$B_{\ell-1} - B'_{\ell-1} = X_o^{k-\ell} (z'_\ell - z_\ell) \ll X_o^{k - \frac{1}{2}\ell}$$

Thus

$$B_{\ell-1} - B'_{\ell-1} \in M(O(X_o^{k-\frac{1}{2}\ell})) .$$

For fixed $B_{\ell-1}$, we have that the number of solutions of $z_1, \ldots, z_{\ell-1}$ is

$$Y_o^{(\frac{1}{2}(\ell-1)\ell - 1)n} X_o^{-(\ell-1)n-n} ,$$

But the solutions of $B_{\ell-1}$ is

$$(X_o^{k-\ell} Y^\ell)^n / X_o^{(k-(\ell-1))n} .$$

Thus the number of solutions $z_1, \ldots, z_\ell$:

$$n(B_r) \ll \left(\frac{X_o^{k-\ell} Y_o^\ell}{X_o^{k-(\ell-1)}}\right)^n Y_o^{(\frac{1}{2}(\ell-1)\ell-1)n} . X_o^{-(\ell-1)n+n} . 1$$

$$\ll Y_o^{(\frac{1}{2}\ell(\ell-1)-1)n} X_o^{-\ell n + n} .$$

LEMMA 6. Let $\tau \geqq 1$, r, r_o, r_1, ℓ be positive rational integers satisfying the conditions :

$$r_o \geqq \ell(\ell-1) + \tau\ell - 1, \qquad k > r = 2r_o = 4r_1 .$$

We define a sum of the following form :

$$A = (X+\sigma_1)^k + \dots + (X+\sigma_{r_o})^k - (X+\sigma_{r_o+1})^k - \dots - (X+\sigma_r)^k$$

$$\|\sigma_i\| < Y = [\sqrt{X}] \qquad (1 \leqq i \leqq r)$$

We denote by $n(A)$ the number of the systems of solutions of $(\sigma_1, \dots, \sigma_r)$ such that

$$A \in M(0(X^{k-\frac{1}{2}(\ell+1)})$$

then we have

$$n(A) \ll Y^{rn + \frac{1}{2}\ell(\ell-1)(1-1/\ell)^{\tau} n} X^{-\frac{1}{2}\ell n}$$

Proof. By the definition of A, we have

$$A = B_\ell + C_\ell$$

$$B_\ell = X^{k-1} U_1 + \dots + X^{k-\ell} U_\ell, \quad C_\ell = X^{k-\ell-1} U_{\ell+1} + \dots + XU_{k-1} + U_k$$

where

$$U_i = \binom{k}{i}(\sigma_1^i + \dots + \sigma_{r_o}^i - \sigma_{r_o+1}^i - \dots - \sigma_r^i) \qquad (1 \leqq i \leqq k).$$

Therefore we have

$$\|U_i\| \leqq r\, k^i\, Y^i \qquad (1 \leqq i \leqq k)$$

From this result and the definition of C_ℓ follows

$$C_\ell \ll X^{k-\frac{1}{2}(\ell+1)}$$

Then by the assumption of B_ℓ we have

$$B_\ell \ll X^{k-1} U_1 + \cdots + X^{k-\ell} U_\ell \in M(O(X^{k-\frac{1}{2}(\ell+1)})).$$

Thus, applying Lemma 5 we obtain the number of $\{U_1, \ldots, U_\ell\}$ is

$$Y^{(\frac{1}{2}\ell(\ell+1)-1)n} X_o^{-\ell n - n}$$

and the number of $B_\ell \in M(O(X^{k-\frac{1}{2}(\ell+1)}))$ is

$$(X^{k-\frac{1}{2}(\ell+1)} / X^{k-\ell})^n$$

Hence the number of $U_1, \ldots, U_\ell$ is

$$\ll (X^{k-\frac{1}{2}(\ell+1)} / X^{k-\ell})^n \, Y^{(\frac{1}{2}\ell(\ell+1)-1)n} X_o^{-\ell n - n}$$

$$\ll Y^{\frac{1}{2}\ell(\ell+1)n} X^{-\frac{1}{2}\ell n}$$

Now, if we put

$$U_1 = z_1, \ldots, U_\ell = z_\ell \qquad ,$$

then applying Lemma 4, the number of $\{\sigma_1, \ldots, \sigma_r\}$ is

$$\ll Y^{nr - \frac{1}{2}\ell(\ell+1)n + \frac{1}{2}\ell(\ell+1)(1-1/\ell)^\tau n}$$

Hence we finally obtain the number $N(A)$ of $\{\sigma_1, \ldots, \sigma_r\}$ with our conditions

$$\ll Y^{nr - \frac{1}{2}\ell(\ell+1)n + \frac{1}{2}\ell(\ell+1)(1-1/\ell)^\tau n} \cdot Y^{\frac{1}{2}\ell(\ell+1)n} X^{-\frac{1}{2}\ell n}$$

$$\ll Y^{nr + \frac{1}{2}\ell(\ell+1)(1-1/\ell)^\tau n} X^{-\frac{1}{2}\ell n}$$

Thus we complete the proof of our Lemma .

Now we write

$$A_h = (X_h + \sigma_{h,1})^k + \cdots + (X_h + \sigma_{h,r_o})^k - (X_h + \sigma_{h,r_o+1})^k - \cdots - (X_h + \sigma_{h,2r_o})^k$$

where

$$\|\sigma_{h,i}\| < Y_h \qquad (0 \leqq h \leqq s_1 - 1,\ 1 \leqq i \leqq r) .$$

Then we have

$$A_h = X_h^{k-1} U_{h,1} + \cdots + X_h U_{h,k-1} + U_{h,k}$$

where

$$U_{h,j} = \binom{k}{j} \eta_{k,j} \quad , \quad \| U_{h,j} \| \leqq r k^i Y_h^j ,$$

$$\eta_{k,j} = \sigma_{h,j}^j + \cdots + \sigma_{h,r_o}^j - \sigma_{h,r_o+1}^j - \cdots - \sigma_{h,2r_o}^j ,$$

$$(1 \leqq j \leqq k) \quad , \quad (0 \leqq h \leqq s_1 - 1)$$

Further we put

$$W = A_o + A_1 + \cdots + A_{s_1 - 1} \qquad (s_1 > 2kn)$$

Using the notations above, we prove the following

LEMMA 7. Let $n(W)$ be the number of the solutions $(\sigma_{o,1}, \ldots, \sigma_{s_1-1,2r_o})$ of the relations

$$W \in M(0\ (X_{s_1-1}^{k - \frac{1}{2}(s_o+1)})) ,$$

then we have

$$n(W) \ll (Y_o Y_1 \ldots Y_{s_1-1})^{rn + \frac{1}{2}s_o(s_o+1)(1-1/s_o)^\tau n} X_o^{-(k-\frac{1}{2})(1-\delta^{s_1})n}$$

$$\ll (Y_o Y_1 \ldots Y_{s_1-1})^{rn} X_o^{\omega}$$

where

$$\omega = \tfrac{1}{2}(s_o + 1)(1 - 1/s_o)^\tau (1 - \delta^{s_1})(k - \tfrac{1}{2})n - (k - \tfrac{1}{2})(1 - \delta^{s_1})n .$$

Proof. By the definition of W we have for $h = 2, \ldots, s_1 - 1$

$$A_h = X_h^{k-1} U_{h,1} + \cdots + U_{h,k}$$

$$\ll X_h^{k-1} Y_h + \cdots + Y_h^k$$

$$(14) \qquad \ll X_{h-1}^{k - \frac{1}{2}(s_o + 1)} \ll X_o^{k - \frac{1}{2}(s_o + 1)} .$$

From these inequalities and the assumption of ours we have

$$A_o = X_o^{k-1} U_{o,1} + \cdots + X_o U_{o,k-1} + U_{o,k} \in M(0(X_o^{k - \frac{1}{2}(s_o + 1)})) .$$

Now the number of solutions of

$$\{ U_{o,1}, \ldots, U_{o,k-1}, U_{o,k} \}$$

is given by the number n_o of solutions of $\{ \sigma_{o,1}, \ldots, \sigma_{o,2r_o} \}$. Since $r_o \geq s_o(s_o + 1) + \tau s_o - 1$, we have by Lemma 6

$$n_o \ll Y_o^{rn + \frac{1}{2}s_o(s_o + 1)(1 - 1/s_o)^\tau n} X_o^{-\frac{1}{2}s_o n} .$$

From (14) and the assumption of Lemma 7, we have

$$W \in M(0(X_1^{k - \frac{1}{2}(s_o + 1)})) .$$

Taking the fixed solution $\{ \sigma_{o,1}, \ldots, \sigma_{o,k} \}$, The number n_1 of solutions $\{ \sigma_{1,1}, \ldots, \sigma_{1,s_o} \}$ of

$$A_1 \in M(0(X_1^{k - \frac{1}{2}(s_o + 1)})$$

is given by

$$n_1 \ll Y_1^{rn + \frac{1}{2} s_o (s_o + 1)(1 - 1/s_o)^{\tau} n} \; X_1^{-\frac{1}{2} s_o n} .$$

Iterating this process, we have finally, by the definition of X_i and δ

$$n(W) \ll (Y_o Y_1 \ldots Y_{s-1})^{rn + \frac{1}{2} s_o (s_o + 1)(1 - 1/s_o)^{\tau} n} \; X_o^{-\frac{1}{2} s_o n} \ldots X_{s_1 - 1}^{-\frac{1}{2} s_o n}$$

$$\ll (Y_o Y_1 \ldots Y_{s_1 - 1})^{rn + \frac{1}{2} s_o (s_o + 1)(1 - 1/s_o)^{\tau} n} \; X_o^{-(1 - \delta^{s_1})(k - \frac{1}{2}) n}$$

whish is the desired result .

For $\xi = \rho_1 x_1 + \cdots + \rho_n x_n$, we write $S(\xi \omega_i) = a_i + d_i$ with rational integers a_i and $-\frac{1}{2} \leqq d_i < \frac{1}{2}$ $(1 \leqq i \leqq n)$ and put

$$\mathfrak{d} = \mathfrak{d}(\xi) = \sum_{i=1}^{n} a_i \rho_i \quad , \qquad \zeta = \zeta(\xi) = \sum_{i=1}^{n} d_i \rho_i$$

where $\mathfrak{d}^{-1} = [\rho_1, \ldots, \rho_n]$ and $\mathfrak{a} = [\omega_1, \ldots, \omega_n]$. We obtain

$$(15) \qquad \xi = \mathfrak{d} + \zeta \quad , \quad \mathfrak{d}^{-1} \mid \mathfrak{a} .$$

LEMMA 8 . We take a point $\xi = (x_1, \ldots, x_n)$ *of* $S(t_o, h_o)$ *which is defined by the Farey division with respect to* (t, h) . *We obtain a pair* α, β *of Lemma 2 with* $1 \mid \alpha$, $\mathfrak{d}^{-1} \mid \beta$.

Let $g_1, \ldots, g_n$ *be integers .* $\Phi^{(i)}$ $(1 \leqq i \leqq n)$ *be positive real numbers with* $\Phi^{(m)} = \Phi^{(m - r_2)}$ $(r_1 + 1 \leqq m \leqq r_1 + r_2)$ *and* ω_o *be a prime number of* $P(R, C)$.

Let $Z(g, \Phi)$ *be the number of prime numbers* ω *of* $P(R, c)$ *with the conditions :*

$$\begin{cases} |\zeta^{(i)}| = \Phi^{(i)} & (1 \leqq i \leqq n) \\ g_m \leqq 4D^{1/n} |\alpha^{(m)}| \, Re\, \zeta^{(m)} < g_m + 1 & (1 \leqq m \leqq n_1 + n_2) \\ g_m \leqq 4D^{1/n} |\alpha^{(m)}| \, Im\, \zeta^{(m)} < g_m + 1 & (r_1 + n_2 + 1 \leqq m \leqq n) \end{cases}$$

where

$$\zeta = \zeta(\xi(\omega^k - \omega_o^k)) .$$

Then we have

$$Z(g, \Phi) \ll |N(\alpha)|^{\Delta} (1 + \prod_{i \in J_1} (h_o \Phi^{(i)} R^{1-k}) \prod_{i \in J_2} (R |\alpha^{(i)}|^{-1}) ,$$

where J_1 *is the set of the indices* i $(1 \leqq i \leqq n)$ *with* $|\alpha^{(i)}| < D^{-\frac{1}{2}}$ *and* J_2 *is the set of other indices .*

Proof . **Hilfssatz 11 of Körner [4] .**

By the classification based upon the Farey division with respect to (t_o, h_o), **we shall prove the Theorem 5 (Vinogradov [14]) .**

Case 1. $\xi \in S(t, h) \cap S(t_o, h_o)$.

By Hölder's inequality we have

$$|R(\xi)|^{r_o} = |R(\xi)|^{2r_1} = \Big| \sum_{\mu \in Q(s_1, X_o)} \sum_{\omega \in P(R, c)} E(\mu \omega^k \xi) \Big|^{2r_1}$$

$$\leqq \sum_{\omega \in P} \Big| \sum_{\mu \in Q} E(\mu \omega^k \xi) \Big|^{2r_1}$$

$$\ll R^{(r_o - 1)n} \sum_{\omega} \Big| \sum_{\mu} E(\mu \omega^k \xi) \Big|^{2r_1}$$

$$(16) \qquad \ll R^{(r_o - 1)n} \sum_{\omega} \Big| \sum_{\mu, \mu' \in Q} E(\xi \omega^k (\mu - \mu')) \Big|^{r_1} = R^{(r_o - 1)n} V ,$$

where

$$V = V(\xi) = \sum_{\omega \epsilon P} \left| \sum_{\mu,\mu' \epsilon Q} E(\xi \omega^k (\mu - \mu')) \right|^{r_1}$$

$$= \sum_{\omega \epsilon P} \sum_{\substack{\mu_1, \dots, \mu_{r_1} \\ \mu'_1, \dots, \mu'_{r_1}}} E(\xi \omega^k (\mu_1 + \dots + \mu_{r_1} - \mu'_1 - \dots - \mu'_{r_1}))$$

For fixed integral χ we put

$$(17) \qquad \chi = \mu_1 + \dots + \mu_{r_1} - \mu'_1 - \dots - \mu'_{r_1} \qquad \mu, \mu' \epsilon Q .$$

Let $\Delta(\chi)$ the number of solutions $\{ \mu_1, \dots, \mu_{r_1}, \mu'_1, \dots, \mu'_{r_1} \}$ of the above equation (17), then we have .

$$V(\xi) = \sum_{\omega} \sum_{\chi} \Delta(\chi) E(\xi \omega^k \chi) = \sum_{\chi} \Delta(\chi) \sum_{\omega} E(\xi \omega^k \chi) .$$

Thus

$$V^2 \leqq |V|^2 = | \Delta(\chi) \sum_{\omega} E(\xi \omega^k \chi) |^2$$

$$\ll \left(\sum_{\chi} \Delta^2(\chi) \right) \left(\sum_{\chi} \left| \sum_{\omega} E(\xi \omega^k \chi) \right|^2 \right)$$

$$(18) \qquad = K \sum_{\omega, \omega_o \epsilon P} \sum_{1 | \chi \epsilon M(X_o^{k-\frac{1}{2}})} E(\xi \chi (\omega^k - \omega_o^k)) .$$

Now we quote the well-known lemma (Hilfssatz 8 of Körner [4]) .

LEMMA 9 . We have

$$\sum_{1 | \lambda \epsilon M(T)} E(\lambda \xi) \ll T^{n-1} Min(T, \| \zeta \|^{-1})$$

where $\zeta = \zeta(\xi)$.

We apply this lemma to the inner sum of (18) we have

$$V^2 \ll K \sum_{\omega, \omega_o \epsilon P} X_o^{(k-\frac{1}{2})(n-1)} Min\,(X_o^{k-\frac{1}{2}}, \| \zeta(\omega^k - \omega_o^k)\xi) \|^{-1}$$

$$= K X_o^{(k-\frac{1}{2})(n-1)} \sum_{\omega_o} \sum_{\omega} Min\,(X_o^{k-\frac{1}{2}}, \| \zeta(\omega^k - \omega_o^k)\ \xi) \|^{-1})$$

$$(19) \qquad = K X_o^{(k-\frac{1}{2})(n-1)} \sum_{\omega_o} R(\xi, \omega_o),$$

where

$$R(\xi, \omega_o) = \sum_{\omega} Min\,(X_o^{k-\frac{1}{2}}, \| \zeta(\omega^k - \omega_o^k)\xi) \|^{-1})$$

$$\leqq \sum_{\substack{\omega \\ \|\zeta\| > X_o^{b-(k-\frac{1}{2})}}} Min\,(X_o^{k-\frac{1}{2}}, \| \|^{-1}) + \sum_{\substack{\omega \\ \|\zeta\| \leqq X_o^{b-(k-\frac{1}{2})}}} Min\,(X_o^{k-\frac{1}{2}}, \| \|^{-1})$$

$$= \sum_{\substack{\omega \\ \|\zeta\| > X_o^{b-(k-\frac{1}{2})}}} X_o^{k-\frac{1}{2}-b} + \sum_{\substack{\omega \\ \|\zeta\| \leqq X_o^{b-(k-\frac{1}{2})}}} X_o^{k-\frac{1}{2}}$$

$$(20) \qquad = S_1 + S_2 .$$

By Lemma 3, we have

$$(21) \qquad S_1 = \sum_{\substack{\omega \\ \|\zeta\| > X_o^{b-(k-\frac{1}{2})}}} X_o^{k-\frac{1}{2}-b} \ll X_o^{k-\frac{1}{2}-b} \frac{R^n}{\log R} \ll R^n X_o^{k-\frac{1}{2}-b}$$

We apply Lemma 8 with

$$(22) \qquad \Phi^{(i)} = X_o^{b-(k-\frac{1}{2})}$$

to estimate S_2. Let Z_1 be the number of all distinct integral n-tuples $g = (g_1, \ldots, g_n)$.

If $\ell \in J_1$, then, by (22) and Lemma 8, we have $\prod_{\ell \in J_1} \ll 1$. Hence we have

$$Z_1 \ll \prod_{i \in J_1} \prod_{i \in J_2} \ll \prod_{i \in J_2} (\,|\alpha^{(i)}|\, X_o^{b-(k-\frac{1}{2})} - 1\,).$$

Thus, by these inequalities

$$S_2 = \sum_{\substack{\omega \in P \\ ||\zeta|| \leqq X_o^{b-(k-\frac{1}{2})}}} X_o^{k-\frac{1}{2}} \leqq X_o^{k-\frac{1}{2}} Z_1 \, Max\, Z(g, X_o^{b-(k-\frac{1}{2})})$$

$$\ll X_o^{k-\frac{1}{2}} |N(\alpha)|^{\varepsilon'} \Big(\prod_{i \in J_2} |\alpha^{(i)}| X_o^{b-(k-\frac{1}{2})} + 1 \Big) \Big(\prod_{i \in J_1} (b_o X_o^{b-(k-\frac{1}{2})} R^{1-n} \Big)$$

$$\prod_{i \in J_2} (R\,|\,\alpha^{(i)}|^{-1} + 1)$$

$$\ll X_o^{k-\frac{1}{2}+\epsilon} + X_o^{k-\frac{1}{2}+\epsilon} (b_o X_o^{b-(k-\frac{1}{2})})^n$$

$$+ X_o^{k-\frac{1}{2}+\epsilon} \prod_{i \in J_1} (b_o X_o^{b-(k-\frac{1}{2})} R^{1-n}) \prod_{i \in J_2} (R\,|\,\alpha^{(i)}|^{-1})$$

$$+ X_o^{k-\frac{1}{2}+\epsilon} \prod_{i \in J_2} (\,|\alpha^{(i)}|\, X_o^{b-(k-\frac{1}{2})}) \prod_{i \in J_1} (b_o X_o^{b-(k-\frac{1}{2})}) \prod_{i \in J_2} (R\,|\alpha^{(i)}|^{-1})$$

$$(23) \qquad = S_{11} + S_{o1} + S_{1o} + S_{oo} \, .$$

We can see easily

$$(24) \qquad S_{11} = X_o^{k-\frac{1}{2}+\epsilon} \ll R^n \, X_o^{k-\frac{1}{2}-3/4}$$

Next we have

$$(25)\qquad S_{o1}=X_o^{k-\frac12+\epsilon}(b_oX_o^{b-(k-\frac12)})^n=R^nX_o^{k-\frac12+\epsilon(b-(k-\frac12))n}R^{(k-(n-1)b+\epsilon)\,n-n}$$

$$=R^nX_o^{k-\frac12-b+(\frac12-1/(2\sqrt{k})\,(k-1+(n-1)b)\,n+\epsilon}=R^nX_o^{k-\frac12-b-\frac14 n}$$

$$(k\geqq 4).$$

For the third sum, we have by the definition :

$$(26)\qquad S_{10}=X_o^{k-\frac12+\epsilon}\prod_{\substack{i\in J_1\\|\alpha^{(i)}|<D^{\frac12}}}(b_oX_o^{b-(k-\frac12)}R^{1-k})\prod_{\substack{i\in J_2\\|\alpha^{(i)}|\geqq D^{\frac12}}}(R|\alpha^{(i)}|^{-1})$$

If $J_1\neq\phi$, we put $\#(J_1)$ the number of elements of J_1. Then we have

$$S_{10}=X_o^{k-\frac12+\epsilon}R^n\prod_{i\in J_1}(b_oX_o^{b-(k-\frac12)}R^{-k})$$

$$\ll R^nX_o^{k-\frac12+\epsilon}X_o^{(b-(k-\frac12)+(1-1/(2\sqrt{k}))\#(J_1)+\epsilon''}$$

$$(27)\qquad \ll R^nX_o^{k-\frac12-\frac12 n}\,.$$

If $J_1=\phi$, then we have

$$S_{o1}=X_o^{k-\frac12+\epsilon}\prod_{i\in J_2}(R|\alpha^{(i)}|^{-1})=X_o^{k-\frac12+\epsilon}R^n\prod_{i\in J_2}|\alpha^{(i)}|^{-1})$$

$$\ll R^nX_o^{k-\frac12+\epsilon}t_o^{-1}\ll R^nX_o^{k-\frac12+\epsilon}R^{-b}\ll R^nX_o^{k-\frac12-(1-1/(2\sqrt{k}))b+\epsilon}$$

$$(28)\qquad \ll R^nX_o^{k-\frac12-\frac12 b}\,.$$

Finally we have for S_{oo} :

$$S_{oo} = X_o^{k-\frac{1}{2}+\epsilon} \prod_{i\epsilon J_2} (|\alpha^{(i)}| X_o^{b-(k-\frac{1}{2})}) \prod_{i\epsilon J_1} (b_o X_o^{b-(k-\frac{1}{2})} R^{1-k}) \cdot \prod_{i\epsilon J_2} (R|\alpha^{(i)}|^{-1})$$

$$\ll R^n X_o^{k-\frac{1}{2}+\epsilon} \prod_{i\epsilon J_2} X_o^{b-(k-\frac{1}{2})} \prod_{i\epsilon J_1} (t_o^{n-1+\epsilon} X_o^{b-(k-\frac{1}{2})})$$

$$(29) \quad \ll R^n X_o^{k-\frac{1}{2}-\frac{1}{2}n} .$$

Thus from (23), (24), (25), (26), (27), (28) and (29)

$$V^2 \ll K X_o^{(k-\frac{1}{2})(n-1)} \sum_{\omega_o} (S_1 + S_2)$$

$$\ll K X_o^{(k-\frac{1}{2})(n-1)} \sum_{\omega_o} (R^n X_o^{k-\frac{1}{2}-b} + S_{11} + S_{o1} + S_{1o} + S_{oo}) ,$$

where

$$(30) \qquad S_2 \ll S_{oo} + S_{o1} + S_{10} + S_{11} \ll R^n X_o^{k-\frac{1}{2}-\frac{1}{2}b} .$$

Hence we have by (19), (20), (21) and (30)

$$(31) \qquad V^2 \ll K X_o^{(k-\frac{1}{2})(n-1)} R^n X_o^{k-\frac{1}{2}-\frac{1}{2}b} \ll K R^{2n} X_o^{(k-\frac{1}{2})n-\frac{1}{2}b} .$$

Now, by the definition of W we have

$$W = A_o + A_1 + \cdots + A_{s_1-1}$$

$$= \mu_1 - \ldots - \mu_{r_1} - \tilde{\mu}_1 - \ldots - \tilde{\mu}_{r_1} - \mu'_1 - \ldots - \mu'_{r_1} - \tilde{\mu}'_1 - \ldots - \tilde{\mu}_{r_1}$$

where

$$\mu_i ; \mu_i , \mu'_i , \mu'_i \in Q(s_1, X_o) \qquad (1 \leq i \leq r_1)$$

Hence K is the number of solutions of $W = 0$ and then we can use the result of Lemma 7 .

Therefore we have by (31)

$$V^2 \ll (Y_o Y_1 \dots Y_{s_1 - 1})^{2n r_o} \cdot$$

$$\cdot X_o^{-(k-\frac{1}{2})(1-\delta^{s_1})n + \frac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\frac{1}{2})n} R^{2n} X_o^{(k-\frac{1}{2})n-\frac{1}{2}b}$$

$$\ll (Y_o Y_1 \dots Y_{s_1})^{2n r_o} R^{2n} X_o^{(k-\frac{1}{2})\delta^{s_1} n + \frac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\frac{1}{2})n - \frac{1}{2}b}$$

Thus, by (16)

$$|R(\xi)|^{r_o} \ll R^{(r_o-1)n} V \ll R^{(r_o-1)n} (Y_o Y_1 \dots Y_{s_1-1})^{nr_o} R^n .$$

$$X_o^{\frac{1}{2}((k-\frac{1}{2})\delta^{s_1} n + \frac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\frac{1}{2})n - \frac{1}{2}b)}$$

$$\ll (Y_o \dots Y_{s_1-1} R)^{nr_o} X_o^{r_o \rho} .$$

Finally we have

$$R(\xi) \ll (Y_o Y_1 \dots Y_{s_1-1} R)^n X_o^{\rho} = Q' X_o^{\rho} ,$$

where

$$Q' = Y_o Y_1 \dots Y_{s_1-1} R$$

and

$$\rho = \frac{1}{2 r_o}((k-\tfrac{1}{2})\,\delta^{s_1} n + \tfrac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\tfrac{1}{2})n - \tfrac{1}{2}b).$$

Case 2. $\xi \in S(t,h) \cap B_\gamma(t_o, h_o)$ of some γ.

By the definition of $B_\gamma(t_o,h_o)$ there is for some γ_o with $\gamma_o \equiv \gamma \mod d^{-1}$, $\gamma_o \to \mathfrak{a}$, $N(\mathfrak{a}) \leqq t_o^n$ and

$$(32) \qquad h_o \| \xi - \gamma_o \| \leqq t_o^{n-1} N(\mathfrak{a})^{-1}$$

From $\xi \in S(t,h)$ and $t_o = R^b < t = X_o^{1-f}$ $\quad (1 < f \leqq \frac{1}{4})$,

we have

$$(33) \qquad \| \xi - \gamma_o \| > h^{-1} N\mathfrak{a}^{1/n}$$

and then we have

$$(34) \qquad | R(\xi) |^{r_o} \ll R^{n(r_o - 1)} V(\xi)$$

where

$$V(\xi) = \sum_\chi \Delta(\chi) \sum_\omega E(\xi \omega^k \chi) = \sum_{\substack{\sigma \bmod \mathfrak{a} \\ (\gamma_o \to \mathfrak{a})}} \sum_\chi \Delta(\chi) \sum_{\substack{\omega \\ \omega \equiv \sigma (\bmod \mathfrak{a})}} E(\xi \omega^k \chi) .$$

$$(35) \qquad = \sum_{\sigma \bmod \mathfrak{a}} V(\sigma, \xi) ,$$

Hence

$$| V(\sigma, \xi) |^2 = | \sum_\chi \Delta(\chi) \sum_{\substack{\omega \\ \omega \equiv \sigma (\bmod \mathfrak{a})}} E(\xi \omega^k \chi) |^2$$

$$= \sum_\chi \Delta(\chi) E(\chi \sigma^k \gamma_o) \sum_{\substack{\omega \\ \omega \equiv \sigma (\bmod \mathfrak{a})}} E(\chi \omega^k \psi) |^2$$

where $\xi = \gamma_o + \psi$.

Therefore we have from 8 and Schwarz's inequality

$$|V(\sigma,\xi)|^2 \leqq \{ \sum_{\chi} |\Delta(\chi) E(\chi\sigma^k\gamma_o)| \sum_{\substack{\omega \\ \omega\equiv\sigma(\mathrm{mod}\ \mathfrak{a})}} E(\chi\omega^k\psi) \ | \}^2$$

$$\leqq \sum_{\chi} \Delta(\chi)^2 \sum_{\substack{\omega,\omega_o \\ \omega\equiv\omega_o\equiv\sigma(\mathrm{mod}\ \mathfrak{a})}} \sum_{\substack{\omega \\ ||\chi||<X_o^{k-\frac{1}{2}}}} E(\chi\omega^k\psi)|^2$$

$$\ll K X_o^{(n-1)(k-\frac{1}{2})} \sum_{\substack{\omega_o \\ \omega_o\equiv\sigma(\mathrm{mod}\ \mathfrak{a})}} \Big(\sum_{\substack{\omega \\ \omega\equiv\omega_o(\mathrm{mod}\ \mathfrak{a})}} \mathrm{Min}\,(X_o^{k-\frac{1}{2}}, ||\zeta(\psi(\omega^k-\omega_o^k)|| ^{-1})\Big)$$

$$(36) \qquad = K X_o^{(n-1)(k-\frac{1}{2})} \sum_{\substack{\omega_o \\ \omega_o\equiv\sigma(\mathrm{mod}\ \mathfrak{a})}} R^*(\xi, \omega_o),$$

where

$$(37) \qquad R^*(\xi,\omega_o) = \sum_{\substack{\omega \\ \omega\equiv\omega_o(\mathrm{mod}\ \mathfrak{a})}} \mathrm{Min}\,(X_o^{k-\frac{1}{2}}, ||\zeta(\psi(\omega^k-\omega_o^k))||^{-1}).$$

Now, from (32) and (4) we have

$$|S(\psi(\omega^k-\omega_o^k)\tau_j)| \ll |S(\xi-\gamma_o)(\omega^k-\omega_o^k)\tau_j)|$$

$$\ll b_o^{-1} t_o^{n-1} N\mathfrak{a}^{-1} R^k = 1/N\mathfrak{a} \to 0 \ (X_o \to \infty)$$

From this result and (15) and the conditions of ω, we have

$$(38) \qquad \psi(\omega^k-\omega_o^k) = \zeta.$$

Let be $P_1 = P_1(R,c)$ the set of all prime numbers in $P(R,c)$ with the conditions

$$\omega \equiv \omega_o \qquad mod\ \mathfrak{a}$$

$$\underset{1 \leqq \ell \leqq n}{Min} |\omega^{(\ell)} - \omega_o^{(\ell)}| \geqq N(\mathfrak{a})^{1/n}$$

and let be $P_2 = P_2(R,c)$ the set all prime numbers ω in P with $\omega \equiv \omega_o$ mod $\mathfrak{a}$, $\omega \notin P_1$.

Let $P_1^*(R,c)$ be the number of elements of the set P_1 and let $P_2^*(R,c)$ be the number of elements of the set $P_2 = P - P_1$. Before the estimations of $P_1^*(R,c)$ and $P_2^*(R,c)$ we shall describe a lemma by Mitsui [6] .

LEMMA 10. Let $\mathfrak{f}$ be a fractional or integral ideal. We take positive numbers $A_1, A_2, \ldots, A_n$ such that $A_{p'} = A_p$ $(n_1 + 1 \leqq p \leqq n_1 + n_2)$ and α_p, β_p $(n_1 + 1 \leqq p \leqq n_1 + n_2)$ such that

$$\beta_p < \alpha_p \leqq 2\pi + \beta_p \qquad (n_1 + 1 \leqq p \leqq n_1 + n_2).$$

We denote by $n(\mathfrak{f}, A, \alpha, \beta)$ the number of the elements ν of $\mathfrak{f}$ satisfying the conditions

$$0 < \nu^{(q)} = A_q \qquad (1 \leqq q \leqq r_1)$$

$$|\nu^{(p)}| \leqq A_p \qquad (n_1 + 1 \leqq p \leqq n_1 + n_2)$$

$$\beta_p \leqq Arg\ \nu^{(p)} \leqq \alpha_p$$

Then we have

$$n(\mathfrak{f}, A, \alpha, \beta) = \frac{1}{D N(\mathfrak{f})} \prod_{j=1}^{n} A_j \prod_{p=\eta_1+1}^{n_1+n_2} (\alpha_p - \beta_p)$$

$$+ 0 \left(\frac{A_o^{n-1}}{N(\mathfrak{f})^{1-1/n}} \right)$$

where $A_o = Max\,(N(\mathfrak{f})^{1/n},\ (A_1 A_2 \dots A_n)^{1/n})$.

Now, If $\omega, \omega_o \in P_1$ **then we have by the restriction on** ω **(4)**

$$(39) \qquad |\,\omega^{(i)k} - \omega_o^{(i)k}\,| \geq c\,R^{k-1}\,|\,\omega^{(i)} - \omega_o^{(i)}\,| \,/\, \log R \qquad (1 \leq i \leq n)$$

If $\omega \in P_1$, **we have from this relations and (33)**

$$\|\zeta\|^{-1} = \|\zeta(\psi(\omega^k - \omega_o^k))\|^{-1} = \|\psi(\omega^k - \omega_o^k)\|^{-1} = \|(\xi - \gamma_o)(\omega^k - \omega_o^k)\|^{-1}$$

$$(40) \qquad \ll b\,N\mathfrak{a}^{-1/n} R^{1-k+\epsilon} N\mathfrak{a}^{1/n} \ll X_o^{k-1-f} R^{1-k+\epsilon}$$

By this inequality and Lemma 10 we have

$$(41) \qquad P_1^*(R, c) \ll R^n / N\mathfrak{a}$$

and

$$(42) \qquad P_2^*(R, c) \ll R^{n-1+b} N\mathfrak{a}^{-1}$$

From (36), (37), (40), (41) and (42), we have

$$R^*(\xi, \omega_o) = \sum_{\substack{\omega \\ \omega \equiv \omega_o (mod\ \mathfrak{a})}} Min\,(X_o^{k-\frac{1}{2}},\ \|\zeta\|^{-1}) \ll \sum_{\omega \in P_2} X_o^{k-\frac{1}{2}} +$$

$$+ \sum_{\omega \in P_1} X_o^{k-1+f} R^{1-k} \ll X_o^{k-1+f} R^{1-k+\epsilon} R^n / N\mathfrak{a} + X_o^{k-\frac{1}{2}} R^{n-1+b} / N\mathfrak{a}$$

$$= R^n X_o^{k-\frac{1}{2}-b} \frac{1}{N\mathfrak{a}} (X_o^{f-\frac{1}{2}+b} R^{1-k+\epsilon} + X_o^{b} R^{b-1})$$

$$\ll R^n \frac{1}{N\mathfrak{a}} X_o^{k-\frac{1}{2}-b} (X_o^{b+f-\frac{1}{2}-(1-1/(2\sqrt{k}))(1-k)+\epsilon}$$

$$+ X_o^{b+(b-1)(1-1/(2\sqrt{k}))}) \ll R^n \frac{1}{N\alpha} X_o^{k-\frac{1}{2}-b} (X_o^{b+\frac{1}{4}-\frac{1}{2}-(1-1/(2\sqrt{k}))(k-1)+\epsilon}$$

$$+ X_o^{b-(1-b)(1-1/(2\sqrt{k}))}) \ll R^n \frac{1}{N\alpha} \cdot X_o^{k-\frac{1}{2}-b} (X_o^{\frac{1}{4}-2(1-1/3)} + X_o^{2b-1-(1-b)/(2\sqrt{9})})$$

$$\ll R^n \frac{1}{N\alpha} X_o^{k-\frac{1}{2}-b-1/30} \tag{43}$$

Thus by (36), (43), we obtain

$$V(\sigma,\xi) \ll K X_o^{(n-1)(k-\frac{1}{2})} \sum_{\substack{\omega_o \\ \omega_o \equiv \sigma(\alpha)}} (R^n \frac{1}{N\alpha} X_o^{k-\frac{1}{2}-b-1/30})$$

$$\ll K R^{2n} \frac{1}{N\alpha^2} X_o^{(k-\frac{1}{2})n-b-1/30}$$

$$\ll ((Y_o Y_1 \ldots Y_{s_1-1})^{r_o} R)^{2n} \frac{1}{N\alpha^2} X_o^{(k-\frac{1}{2})n-b-1/30} .$$

$$. X_o^{\frac{1}{2}(s_o+1)(1-1/s_o)^{\tau} (1-\delta^{s_1})(k-\frac{1}{2})n-(k-\frac{1}{2})(1-\delta^{s_1})n}$$

$$= ((Y_o Y_1 \ldots Y_{s_1-1})^{r_o} R)^{2n} \frac{1}{N\alpha^2} . \tag{44}$$

$$. X_o^{(k-\frac{1}{2})\delta^{s_1} n+\frac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\frac{1}{2})n-b-1/30} .$$

Finally we have, by (34), (35) and (44)

$$R(\xi)^{r_o} \ll R^{n(r_o-1)} V(\xi) \ll R^{n(r_o-1)} \sum_{\sigma \bmod \alpha} V(\sigma,\xi)$$

$$\ll R^{n(r_o-1)} N\alpha ((Y_o Y_1 \ldots Y_{s_1-1})^{r_o} R)^n \frac{1}{N\alpha}$$

$$X_o^{\frac{1}{2}((k-\frac{1}{2})\delta^{s_1} n+\frac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\frac{1}{2})n-b-1/30))}$$

Therefore

$$R(\xi) \ll (Y_o Y_1 \dots Y_{s_1-1} R)^n X_o^{\frac{1}{2r_o}((k-\frac{1}{2})\delta^{s_1} n + \frac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\frac{1}{2})n-b-1/30)}$$

$$\ll (Y_o Y_1 \dots Y_{s_1} R)^n X_o^{-\frac{1}{2r_o}((k-\frac{1}{2})\delta^{s_1} n - \frac{1}{2}(s_o+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\frac{1}{2})n-\frac{1}{2}b)}$$

so the proof of Theorem 5 is completed.

§ 4. *Proof of the main theorem* .

Let ν be a totally positive integer in J_k. We write $A = \sqrt[n]{N(\nu)}$. If we take a totally positive unit ε and $\Delta (0 < \Delta < \frac{1}{2})$ appropriately, and put $\nu_o = \nu \varepsilon^k$, then we can make

$$\text{(45)} \quad \begin{cases} \dfrac{c_1 A}{\Delta^{2n_2}} < \nu_o^{(\ell)} < \dfrac{c_2 A}{\Delta^{2n_2}} & (1 \leqq \ell \leqq n_1) \\[2ex] c_1 \Delta^{n_1} A < | \nu_o^{(m)} | < c_2 \Delta^{n_2} A & (n_1 + 1 \leqq m \leqq n_1 + n_2) \end{cases}$$

where c_1, c_2 are constants chosen suitably. It should be noticed that c_1 and c_2 can be taken independently of Δ .

We use the following notations :

$$s_o = [\tfrac{1}{2} \log k]$$

$$\text{(46)} \quad s_1 = [2k(6 + \log(n^2 + 1))]$$

$$s_2 = n[k \log k + 3 \log\log k + (2 \log n + \log\log n + 4)]$$

$$T = \left(\frac{c_2 A}{\Delta^{2n_2}}\right)^{1/k} = T'^{1/k}, \text{ where } T' = \frac{c_2 A}{\Delta^{2n_2}}$$

(47) $$T = \left(\frac{c_1 \Delta^{n_1} A}{4s_2 + 2(2c_3)^k s_1}\right)$$

$$X_o = [\sqrt{P}]$$

$$[\tilde{T}] = P$$

Let

(48) $$\varphi = \nu_o - \sigma_1 - \sigma_2 - \mu\omega^k,$$

where

$$\sigma_1, \sigma_2 \in Q(s_2, T), \quad \omega \in P(R, c_3), \quad \mu \in Q(s_1, X_o).$$

Then, by (45), (46), and (47), we have

$$|\varphi^{(i)}| = |\sigma_1^{(i)} + \sigma_2^{(i)} + \mu^{(i)} \omega^{(i)k}|$$

$$< 2s_2 \tilde{T}^k + 2^k s_1 X_o^k c_3^k X_o^{k(1-1/(2\sqrt{k}))}$$

$(1 \leq i \leq n)$.

$$< \tfrac{1}{2} c_1 \Delta^{n_1} A,$$

Moreover we have

(49) $$|\varphi^{(\ell)}| < \frac{c_2 A}{2\Delta^{2n_2}}$$

and

$$(50) \qquad |\varphi^{(\ell)}| > \frac{c_1 A}{\Delta^{2n_2}} - \tfrac{1}{2} c_1 \Delta^{n_1} A > \frac{2-\Delta^{2n_2}}{2\Delta^{2n_2}} c_1 A > \frac{c_1}{\Delta^{2n_2}}$$

$$(1 \leqq \ell \leqq n_1)$$

Finally we have

$$(51) \qquad |\varphi^{(m)}| < c_1 \Delta^{n_1} A + 2 s_2 \tilde{T}^{k} + 2^k s_1 X_o^k c_3^k X_o^{k(1-1/(2\sqrt{k}))}$$

$$\leq c_2 \Delta^{n_1} A + (2 s_2 + (2c_3)^k s_1) \tilde{T}^{k}$$

$$= (c_2 + \tfrac{1}{2} c_1) \Delta^{n_1} A$$

and

$$(52) \qquad |\varphi^{(m)}| \geq c_1 \Delta^{n_1} A - \tfrac{1}{2} c_1 \Delta^{n_1} A = \tfrac{1}{2} c_1 \Delta^{n_1} A > 0$$

We define the following trigonometrical sums :

$$(53) \qquad \begin{cases} L(\xi) = \displaystyle\sum_{\lambda \in \Omega(T)} E(\lambda^k \xi), \\ V(\xi) = \displaystyle\sum_{\mu_o \in Q_o(s_2, T)} E(\mu_o \xi) \\ R(\xi) = \displaystyle\sum_{\mu \in Q(s_1, X_o)} \sum_{\omega \in P(R, c_3)} E(\mu \omega^k) \end{cases}$$

where

$$s_1 > 4nk + 1 .$$

Now we divide the integral

$$\int_U L^{4nk+1}(\xi)\, V^2(\xi)\, R(\xi)\, E(-\nu\epsilon^k \xi)\, dx$$

into two parts by the Farey division $B_\gamma(t,h)$ and $S(t,h)$

$$= \sum_{\gamma\epsilon\Gamma} \int_{B_\gamma} L^{4nk+1}(\xi)\, V^2(\xi)\, R(\xi)\, E(-\nu\epsilon^k \xi)\, dx$$

$$+ \int_S L(\xi)^{4nk+1} V^2(\xi)\, R(\xi)\, E(-\nu\epsilon^k \xi)\, dx$$

where $h = X_o^{k-1+f}$, $t = X_o^{1-f}$ ($f = \frac{1}{4}$).

By the definition of L, V, R, (48) and (53), we have

$$(54)\quad \sum_{\gamma\epsilon\Gamma} \int_{B_\gamma} L^{4nk+1}(\xi)\, V^2(\xi)\, R(\xi)\, E(-\nu\epsilon^k \xi)\, dx$$

$$= \sum_{\gamma\epsilon\Gamma} \int_{B_\gamma} L(\xi)^{4nk+1} \sum_{\varphi} E(-\varphi\xi)\, dx$$

$$= \sum_{\varphi} \sum_{\gamma} \int_{B} L^{4nk-1}(\xi)\, E(-\varphi\xi)\, dx .$$

Here we put $\mu = \varphi / T^k$, then we have by (49), (50), (51), (52) and (48)

$$0 < \frac{c_1}{2c_2} < \tau^{(\ell)} / T^k < \frac{c_2 A}{\Delta^{2n_2} T^k} < 1$$

$$|\mu^{(m)}| < \frac{(c_2 + \frac{1}{2}c_1)\,\Delta^{n_1} A}{(c_2 A / \Delta^{2n_2})} = (1 + \tfrac{1}{2}c_1/c_2)\,\Delta^{n} \quad (n_1 + 1 \leqq m \leqq n_1 + n_2) .$$

Hence by Theorem 2

$$J(\mu) = D^{\frac{1}{2}(1-s)} \prod_{\ell=1}^{n_1} F(\mu^{(\ell)}) \prod_{m=n_1+1}^{n_1+n_2} H(\mu^{(m)}),$$

with

$$F(\mu^{(\ell)}) = \frac{\Gamma(1+1/k)}{\Gamma(s/k)} (\mu^{(\ell)})^{s/k-1} > c_3$$

$$H(\mu^{(m)}) \to H(0) > c_4 \qquad (\Delta \to 0).$$

Hence, if we take Δ sufficiently small, then

$$(55) \qquad J(\mu) > c_5 > 0.$$

It follows, from (54), (55) and Theorem 1, that

$$\mathcal{R}e \sum_{\gamma \in \Gamma} \int_{B_\gamma} L(\xi)^{4nk+1} V(\xi)^2 R(\xi) E(-\nu \epsilon^k \xi)\, dx > c\, Q_o^*(s_2, T)\, Q' T^{(s-k)n}$$

for sufficiently large T.

Furthermore we have

$$\int_S = \int_S L(\xi)^{4nk-1)} V^2(\xi) R(\xi) E(-\nu \epsilon^k \xi)\, dx$$

$$\ll \sup_{(x_1,\ldots,x_n)\in S} |L(\xi)|^{4nk+1} |R(\xi)| \int_U |V(\xi)|^2 dx$$

$$= \sup_{(x_1,\ldots,x_n)\in S} |L(\xi)|^{4nk+1} |R(\xi)|\, Q_o^*(s_2, \bar{T})$$

Now we have to use Theorem 5 and trivial estimation $|L^{s}(\xi)| \ll T^{ns}$. From these results, we have

$$\int_S \ll T^{sn} Q_o^*(s_2, T)\, Q'\, X_o^{\rho} \ll T^{sn} Q_o^*(s_2, T) Q' X_o^{\rho}\, Q_o^*(cs_2, T)\, T^{-nk+nk(1-1/k)^{\ell}}$$

$$\ll T^{(s-k)n} (Q_o^*(s_2, T))^2\, Q'\, X_o^{\rho+nk(1-1/k)^{\ell}} \tag{56}$$

where

$$\ell = [\, s_2/n \,] .$$

Thus finally we have

$$\int_S \ll T^{(s-k)n} (Q_o^*(s_2 T))^2\, Q'\, X_o^{\sigma} \tag{57}$$

where

$$\sigma = \frac{1}{2r_o}\left((k-\tfrac{1}{2})^{s_1} + \tfrac{1}{2}(s_2+1)(1-1/s_o)^{\tau}(1-\delta^{s_1})(k-\tfrac{1}{2}) - \tfrac{1}{2} b/n\right) n$$

$$+ 2nk\,(1-1/k)^{\ell}$$

We have used the following Tatuzawa's Lemma [10] to give (56) .

LEMMA 11. Let $Q_s(T)$ denote the set of integers μ of K which can be expressed in the form

$$\mu = \sigma_1^k + \cdots + \sigma_s^k$$

where σ_r $(1 \leqq r \leqq s)$ is a totally non-negative integer of K with $||\sigma_r|| \leqq T$.

Let $Q_s^*(T)$ be the number of integers belonging to $Q_s(T)$, then,

$$Q_s^*(T) \geqq c\,T^{nk(1-(1-1/k)^{\ell})}$$

where $\ell = [\, s/n \,]$.

If we can prove

$$\sigma > -c\,, \quad c > 0$$

in (57), then we have from (56) and (57)

$$\int_U = \int_U L^s(\xi)\, V^2(\xi)\, R(\xi)\, E(-\nu\epsilon^k \xi)\, dx > 0\,,$$

where $N(\nu) > c$.

This means : there exist total non-negative integers $\lambda_j\ (1 \leqq j \leqq s)$ $\sigma_j\ (1 \leqq j \leqq 2s_2)$ **and** $\tau_j\ (1 \leqq j \leqq s_1)$ **such that**

$$\nu\epsilon^k = \sum_{j=1}^{s} \lambda_j^k + \sum_{j=1}^{2s_2} \sigma_j^k + \sum_{j=1}^{s_1} \tau_j^k$$

with

$$\begin{cases} N(\lambda_j) \leqq T^n = c_7 N(\nu)^{1/k}, \\ N(\sigma_j) \leqq \tilde{T}^n = c_8 N(\nu)^{1/k}, \\ N(\tau_j) \leqq X_o^n R^n = X_o^n (X_o^{1-1/(2\sqrt{k})})^n \leqq c_9\, N(\nu)^{1/k} . \end{cases}$$

Let

$$\frac{\log k + \log(s(n^2+1))}{-\log(1 - s_o/(k-\frac{1}{2}))} = D_1 \; .$$

Since $s_o = [\, \frac{1}{2} \log k \,]$ **and well known inequalities**

Then, we have, by (58) and $s_o = [\,\frac{1}{2}\, logk\,]$,

$$D_\tau < (logk + loglogk + log\,(n^2 + 1) + 1.5\,)\, s_o$$

$$< (\, logk + loglogk + log\,(n^2 + 1) + 2\,)\, \frac{logk}{2}$$

Hence, we have

$$(60) \qquad \tau \geqq D_\tau = \frac{logk + loglogk + log\,(n^2 + 1) + 4}{-\, log\,(1 - 1/s_o)}$$

where

$$(61) \qquad \tau \geqq [\, logk + loglogk + log\,(n^2 + 1) + 2)\frac{logk}{2}\,]\,.$$

Finally we have from (60)

$$\tfrac{1}{2}\, k\, logk\, (1 - 1/s_o)^\tau < \frac{1}{8\,(n^2 - 1)}\,.$$

Thus by this inequality

$$\tfrac{1}{2}\,(s_o + 1)\,(1 - 1/s_o)^\tau\,(1 - \delta^{s_1})\,(k - \tfrac{1}{2})$$

$$(62) \qquad < \tfrac{1}{2}\, klogk\,(1 - 1/s_o)^\tau < \frac{1}{8}\,\frac{1}{n^2 + 1}\,.$$

Hence we have by (59) and (62)

$$(63) \qquad \sigma' = \tfrac{1}{2}\,\sigma < \frac{-b}{16\, r_o} + nk\,(1 - 1/k\,)^{[\frac{s_2}{n}]}\,.$$

We put

$$(64) \qquad r = 2r_o = 2\,[\, 2\, logn\, (logk\,)^3\,]\,.$$

Then we have by the definition of (61)

$$s_o(s_o + 1) + \tau s_o$$

$$< \tfrac{1}{2} \log k (\tfrac{1}{2} \log k + 1 + \tfrac{1}{2} (\log k)^2 + \tfrac{1}{2} \log k \log\log k + \tfrac{1}{2} \log (n^2 + 1) \log k + \log k)$$

$$< \tfrac{1}{2} \log k ((\log k)^2 + (\tfrac{1}{2} \log (n^2 + 1) + 3/2) \log k + 1)$$

$$< 2 \log n \, (\log k)^2$$

$$< 2 r_o \; .$$

Let

$$D_2 = \frac{\log k + 3 \log\log k + \log (32 (n^2 + 1)) + \log\log n}{- \log (1 - 1/k)} \; .$$

Then we have by (58) and the definition of s_2

$$D_2 < k (\log k + 3 \log\log k + 2 \log n + \log\log n + 4) - 1$$

hence we obtain

$$[\frac{s_2}{n}] \geqq k (\log k + 3 \log\log k + 2 \log n + \log\log n - 4)$$

and

$$(65) \qquad (1 - 1/k)^{[\frac{s_2}{n}]} < \frac{1}{32(n^2 + 1) \log n \; k \log^3 k}$$

We have by (64)

$$(66) \qquad \frac{b}{32 r_o} > \frac{1}{32(n^2 + 1) \log n \; k \log^3 k}$$

(58) $$p-1 \leqq \frac{1}{-\log(1-1/p)} \leqq p \qquad (0 < 1/p < 1),$$

we have

$$D_1 < (\log k + \log(8(n^2+1)))\frac{k-\frac{1}{2}}{s_o}$$

$$< (\log k - 2 + \log(8(n^2+1)))\frac{k}{\frac{1}{2}\log k - 1}$$

$$< 2k\left(1 - \frac{\log(8e^2(n^2+1))}{\log k - 2}\right)$$

$$< 2k(1 + \log(n^2+1) + 4.079\ldots)$$

$$< 2k(6 + \log(n^2+1))$$

$$< s_1$$

where

$$s_1 = [2k(6 + \log(n^2+1))] + 1.$$

Thus we have

$$s_1 > D_1 = \frac{\log k + \log(8(n^2+1))}{\log(1 - s_o/(k-2))}$$

and we have

(59) $$(k-\tfrac{1}{2})\,\delta^{s_1} < \frac{1}{8(n^2+1)}.$$

Next, let

$$D_\tau = \frac{\log k + \log\log k + \log(n^2+1) + \log 4}{-\log(1 - 1/s_o)}$$

By (65) and (66)

$$(1-1/k)^{[s_2/n]} < \frac{1}{32(n^2+1)\log n\, k \log^3 k}.$$

and

$$(67) \qquad (1-1/k)^{[s_1/n]} < \frac{b}{32 r_o}.$$

Thus we have by (63) and (67)

$$\sigma' = \tfrac{1}{2}\sigma < \frac{-b}{16 r_o} + nk\,(1-1/k)^{[s_2/n]} < -\frac{b}{32 r_o} < 0$$

Hence we have $\int_U > 0$. Finally we have

$$G_K(k) \leqq s + s_1 + 2s_2$$

$$\leqq 4nk + 1 + 2k\,(6 + \log(n^2+1))$$

$$+ 2nk\,(\log k + 3\log\log k + (2\log n + \log\log n + 4))$$

$$\leqq 2nk\log k + 6k\log\log k$$

$$+ 2\,(\log n + n\log\log n + 2\log n\ 14)\,k + 1 .$$

Our Main Theorem follows from this result.

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Department of Mathematics

Nagoya Institute of Technology

Gokisa Showa-ku, Nagoya, 466, Japan

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