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ALGEBRAS OF INTEGRALS

by

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and

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ABSTRACT

A classical theorem in mechanics states that a Hamiltonian which is invariant under a symmetry group admits additional integrals of motion. This paper investigates the converse of the above theorem. If a Hamiltonian admits integrals then a symmetry can be constructed and the flow studied on a quotient space. The quotient space is shown to be symplectic and the resulting flow Hamiltonian. The constructions used are similar to the recent constructions of Nehoroshev, Marsden and Weinstein and Meyer.

The general theory presented is used to give an intrinsic derivation of Hamilton's equations of motion. Also special local coordinates are given which display the integrals in a simple form.

Forward. **Miss Uei Fong died suddenly before completing her doctoral research. From her notes, I was able to complete her dissertation and she was posthumously awarded the degree of Doctor of Philosophy by the University of Cincinnati in June**

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1974. Her thesis along with several improvements and extensions is presented in this paper.

In Miss Fong's behalf, I would like to thank the Charles Phelps Taft foundation for granting her a fellowship during her studies at the University of Cincinnati. K. R. M.

1. *Introduction :* A classical theorem in mechanics states that a Hamiltonian which is invariant under the action of a group admits additional integrals. In [7] Smale shows that in this case the flow defined by the Hamiltonian restricted to an integral surface is invariant under a subgroup of the full group and so the flow is naturally defined on the quotient space obtained by identifying orbits of the subgroup in an integral surface. He uses this reduction to investigate relative equilibrium points in the planar n-body problem. Subsequently, Nehoroshev [6], Marsden and Weinstein [3], and Meyer [4] have shown that the resulting quotient space is symplectic and the resulting flow is Hamiltonian. This result is a generalization of a theorem of Reeb [9] which states that the orbit space of a Hamiltonian flow restricted to an energy surface naturally carries a symplectic structure (also see Souriau [8]). In this case the group is the flow itself.

This paper investigates Hamiltonian systems which admit additional integrals but an apriori group action is not given. When the Hamiltonian system admits additional integrals we again construct a quotient space where the flow can be studied. This gives a global generalization of the classical reduction of a Hamiltonian system of n-degrees of freedom to a Hamiltonian system of $n-k$ degrees of freedom when k integrals in involution are known. These general results are developed in section 2.

The third section discusses particular situations in view of the general theory. In this section we recoup the theorem on systems with symmetries, give special local coordinates, give an intrinsic derivation of Hamilton's equations of motion and discuss the characteristic multipliers of a periodic orbit in the presence of integrals. Our derivation of Hamilton's equations of motion is novel in that it does not require the Lagrangian formulation as a starting point. The derivation proceeds directly from the Newtonian formulation to the Hamiltonian formulation.

2. *Notation and General Results.* Throughout this paper all manifolds, functions, forms, etc. will be C^∞. The notation of symplectic geometry used here follows closely the notation given in [1]. The reader is referred to [1] for the basic theory of symplectic manifolds and to [5] for the basic theory of distributions. Let M be a symplectic manifold of dimension $2n$ with symplectic structure Ω, ie. Ω is a closed, nondegenerate 2-form on M. Thus for $m \in M$, Ω_m is a nondegenerate skew symmetric bilinear form on T_mM and so T_mM is a symplectic linear space with symplectic inner product Ω_m. The symplectic inner product Ω_m defines an isomorphism $\flat : T_mM \to T_m^*M : v \to v^\flat = \Omega_m(v,.)$. Let $\# : T_m^*M \to T_mM : v \to v\#$ be the inverse of $\flat$. Let $\mathcal{F} = \mathcal{F}(M)$ denote the smooth real valued functions on M, $\mathcal{X} = \mathcal{X}(M)$ the smooth vector fields on M and $\mathcal{X}^* = \mathcal{X}^*(M)$ the smooth one forms on M. The symplectic structure Ω defines the Poisson bracket operator $\{\, ,\, \}$ and turns both $\mathcal{F}$ and $\mathcal{X}^*$ into Lie algebras. If $H \in \mathcal{F}$ then $dH \in \mathcal{X}^*$ and $dH^\# = (dH)^\# \in \mathcal{X}$. $dH^\#$ is called the Hamiltonian vector field whose Hamiltonian is H.

For any linear space V the dual will be denoted by V^*. If $U \subset V$ then let $U^\circ = \{f \in V^* : f(U) = 0\}$ and if $U \subset V^*$ then let $U^\circ = \{u \in V : f(u) = 0$ for all $f \in U\}$.

Before proceeding with the formal development, consideration of the following situation will help motivate the definitions and lemmas to follow. Let $H \in \mathcal{F}$

then the Hamiltonian vector field $dH^{\#}$ defines a flow on M. One method for analyzing the flow defined by $dH^{\#}$ is to find all the global integrals for the flow and then study the restriction of the flow on the invariant level sets of these integrals. In general these invariant sets may not be manifolds and even if they are manifolds they may not carry a symplectic structure. However, under some mild nondegeneracy assumptions it will be shown that a quotient space of these invariant sets does admit a symplectic structure and a naturally defined flow on this quotient space is Hamiltonian. The set of all integrals for $dH^{\#}$ is the annihilating subalgebra of H in $\mathfrak{F}$, i.e. the set of all integrals of $dH^{\#}$ is $\mathfrak{I} = \mathfrak{I}(H) = \{ F \in \mathfrak{F} : \{ H, F \} = 0 \}$ and $\mathfrak{I}$ is a subalgebra of $\mathfrak{F}$. In many physical examples $\mathfrak{I}$ is not known completely but a subalgebra of $\mathfrak{I}$ is known.

LEMMA 1. Let V be a symplectic linear space with symplectic inner product Ω and $W \subset V^*$ a subspace. Then $W^{\circ} / (W^{\#} \cap W^{\circ})$ admits a natural symplectic inner product ω defined by $\omega([x], [y]) = \Omega(x, y)$.

Proof: The proof given here is the same as found in [3] or [4]. If $v \in W^{\#}$ and $u \in W$, then $\Omega(v, u) = 0$ by definition. Thus if $x, y \in W^{\circ}$ and $\xi, \eta \in W^{\#} \cap W^{\circ}$ then $\Omega(x + \xi, y + \eta) = \Omega(x, y)$ and so ω is well defined. If $\omega([w], [y]) = 0$ for all $[y] \in W^{\circ} / (W^{\#} \cap W^{\circ})$, then $\Omega(x, y) = 0$ for all $y \in W$ or $\Omega(x, .) \in W$. Thus $x \in W^{\#}$ or $[x] = 0$. This shows that ω is nondegenerate on $W^{o} | (W^{\#} \cap W^{o})$. Clearly ω is skew symmetric.

In order to make a global construction based on the above, some notation must be given. Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{F}$, $S_m = S_m(\mathfrak{A}) = \{ dF(m) \in T^*_m M : F \in \mathfrak{A} \}$, $S = S(\mathfrak{A}) = \bigcup_{m \in M} S_m$. Let $S^0 = \bigcup_{m \in M} S^0_m$, $S^{\#} = \bigcup_{m \in M} S^{\#}_m$ and $S^{\#} \cap S^0 = \bigcup_{m \in M} (S^{\#}_m \cap S^0_m)$. Clearly S_m is a linear subspace of $T^*_m M$ and if $\dim S_m$

is independent of m then S is a smooth sub-bundle of T^*M. The algebra $\mathfrak{A}$ will be said to have rank q if $\dim S_m = q$ for all $m \in M$. If $\mathfrak{A}$ has rank q then S is an involutive distribution of M of rank $2n-q$ (see [5] Theorem 2. 11.11).

LEMMA 2: Let $\mathfrak{A}$ be a subalgebra of $\mathfrak{F}$ of rank q. Then $S^{\#}$ is an involutive distribution and if $\dim(S^{\#}_m \cap S^{\circ}_m)$ is independent of $m \in M$ then $S^{\#} \cap S^{\circ}$ is an involutive distribution.

Proof: Let $m_0 \in M$ and $f_1, \ldots, f_q \in \mathfrak{A}$ such that $df_1(m_0), \ldots, df_q(m_0)$ span S_{m_0}. Then there is an open neighborhood O of m_0 such that $df_1(m), \ldots, df_q(m)$ span S_m for all $m \in O$. Let $X_1 = df_1^{\#}, \ldots, X_q = df_q^{\#}$ so $X_i \in \mathfrak{X}$ and $X_1(m), \ldots, X_q(m)$ span $S^{\#}_m$ for all $m \in O$. Now $[X_i, X_j] = \{df_i, df_j\}^{\#} = d\{f_i, f_j\}^{\#}$. Since $\mathfrak{A}$ is an algebra $\{f_i, f_j\} \in \mathfrak{A}$ and so $d\{f_i, f_j\}(m) \in S_m$. Thus $[X_i, X_j](m) \in S^{\#}_m$ for all $m \in O$. Thus $S^{\#}$ is an involutive distribution. Clearly if the intersection of two involutive distributions has constant dimension then the intersection is an involutive distribution. Thus the second part of the lemma is now obvious.

Since S^0 is an involutive distribution for each $m_0 \in M$ there is a unique maximal connected integral manifold N of S^0 through m_0. That is, $m_0 \in N$, N is a connected submanifold of M, $T_m N = S^0_m$ for all $m \in N$ and N is maximal with respect to these properties. Let $m_0 \in M$ be fixed and N the maximal integral manifolds of S^0 through m_0. Since $S^{\#} \cap S^0$ is an involutive distribution on M and $S^{\#}_m \cap S^0_m \subset T_m N$ for each $m \in N$ one can consider $S^{\#} \cap S^0$ as an involutive distribution on N. Let $\widetilde{\mathfrak{S}} = (S^{\#} \cap S^0) \mid N = \bigcup_{m \in N} (S^{\#}_m \cap S^0_m)$. For each $m \in N$ let L_m be the maximal integral manifolds of $\widetilde{\mathfrak{S}}$

in N. If m and $m' \in N$ then define $m \sim m' \in L_m$. Clearly $\sim$ is an equivalence relation on N. Let B be the quotient space of N modulo this equivalence relation and $\pi : N \to B$ the projection map. In general B may not be a manifold so some additional assumptions must be made.

THEOREM 1. Let $\mathfrak{A}$ *be a subalgebra of* $\mathfrak{F}$ *of rank* q *and* $m_0 \in N \subset M$ *as above. Let* $\dim (S^{\#}_m \cap S^0_m)$ *be independent of* $m \in N$ *so that* $\pi : N \to B$ *is defined. If* B *is a manifold and* $\pi : N \to B$ *a fiber bundle then* B *is a symplectic manifold with symplectic structure* ω. *Moreover,* $\Omega \mid N = \pi^* \omega$ *and if* $\pi(m) = b$ *then* $D\pi : T_m N = S^0_m \to T_b B$ *has kernel* $S^{\#}_m \cap S^0_m$.

Remark: For any $m_0 \in M$ there is an open neighborhood O of m_0 such that $N \to B$ is a trivial disk bundle over a disk (see [6], Theorem 2.11.8).

Proof.: Let $m \in N$ be such that $\pi(m) = b$. Then $D\pi : T_m N = S^0_m \to T_b B$ is surjective and has kernel $S^{\#}_m \cap S^0_m$ by construction. Thus $T_b B$ is isomorphic to $S^0_m / (S^{\#}_m \cap S^0_m)$ and so by lemma 1 the space $T_b B$ has a symplectic inner product. However, this inner product must be shown to be independent of $m \in \pi^{-1}(b)$. Since N is connected it is enough to show that for each $m \in \pi^{-1}(b)$ there is a neighborhood O of m such that the symplectic inner product defined by $S^0_m / (S^{\#}_m \cap S^0_m)$ and $S^0_r / (S^{\#}_r \cap S^0_r)$ on $T_b B$ are the same for all $r \in O$. This will be shown by constructing a symplectic isomorphism $\psi : T_m M \to T_r M$ such that $\psi(S^0_m) = S^0_r$ and $\psi(S^{\#}_m \cap S^0_m) = S^{\#}_r \cap S^0_r$. This symplectic isomorphism will be constructed as the derivative of the time one map of a Hamiltonian vector field on M which leaves N and L_m invariant.

Let the dimension of N be $\alpha = 2n - q$ and the dimension of L_m be β. Let U be an open neighborhood of m in M and $x_1, \ldots, x_{2n}$ a coordinate system at m such that 1) m has coordinates $x_1 = \ldots = x_{2n} = 0$, 2) U is given by $|x_i| < 1$ for $i = 1, \ldots 2n$, 3) $N \cap U$ is given by $|x_i| < 1$ for $i = 1, \ldots, \alpha$ and $x_{\alpha+1} = \ldots = x_{2n} = 0$, and 4) the leaves of

S in $N \cap U$ are given by $|x_i| < 1$ for $i = 1, \ldots, \beta$, $x_i = a_i$ for $i = \beta + 1, \ldots, \alpha$ (a_i's constant such that $|a_i| < 1$) and $x_i = 0$ for $i = \alpha + 1, \ldots, 2n$. Such a coordinate system exists by theorem 2.11.8 of [5] but it need not be symplectic. The neighborhood 0 is given by $0 = L_m \cap U$. In these coordinates $L_m \cap U = 0$ is given by $|x_i| < 1$ for $i = 1, \ldots, \beta$ and $x_i = 0$ for $i = \beta + 1, \ldots, 2n$. Let $r \varepsilon 0 = L_m \cap U$ have coordinates $x_i = b_i$ for $i = 1, \ldots, \beta$. The vector field $Y = b_i \, \partial / \partial x^1 + \cdots + b_\beta \, \partial / \partial x^\beta$ will be considered as defined on $N \cap U$. As such it is clear that Y is tangent to N and to the leaves of S. Moreover the solution of Y through m at $t = 0$ passes through r when $t = 1$.

Let $\theta = Y^b$ so θ is a one form defined on $N \cap U$ - - however it may take values in the cotangent bundle of $M \cap U = U$. Thus we consider $\theta : N \cap U \to T^* U$ with the usual projection property. If X is any vector field on M which is tangent to N then $\theta_s (X_s) = \Omega_s (Y_s, X_s) = 0$ for all $s \varepsilon N \cap U$ since $X_s \varepsilon S_s^0$ and $Y_s \varepsilon S_s^\# \cap S_s^0$. Thus in the above coordinates

$$\theta = \sum_{i=\alpha+1}^{2n} \theta_i (x_1, \ldots, x_\alpha) \, dx^i$$

Define $G : U \to R$ by the formula

$$G(x_1, \ldots, x_{2n}) = \sum_{i=\alpha+1}^{2n} \theta_i (x_1, \ldots, x_\alpha) \, x^i$$

so that $dG \,|\, (N \cap U) = \theta$ or $Y = dG^\# | N$.

Thus we have extended the vector field Y which was only defined on $N \cap U$ to a Hamiltonian vector field $dG^\#$ defined on U. By construction $dG^\#$ is tangent to N and L_m and the solution of $dG^\#$ through m at $t = 0$ passes through r at $t = 1$. Let ϕ_t be the flow defined by $dG^\#$ so $\phi_1(m) = r$

$D\phi_1 : T_m M \to T_r M$ is symplectic, $D\phi_1(S^0_m) = S^0_r$ and $D\phi_1(S^{\#}_m \cap S^0_m) = S^{\#}_r \cap S^0_r$. Thus $\psi = D\phi_1$ is the desired symplectic isomorphism.

THEOREM 2: Let the notation and hypotheses of Theorem 1 hold. Let $H \in \mathfrak{F}$ be such that $\{H, \mathfrak{A}\} = 0$, ie., $\{H, f\} = 0$ for all $f \in \mathfrak{A}$. Thus each $f \in \mathfrak{A}$ is an integral of $dH^{\#}$ and N is an invariant manifold for the flow defined by $dH^{\#}$. Then H is constant on $\pi^{-1}(b)$ for all $b \in B$ and so $h \in \mathfrak{F}(B)$ may be defined by $h = H.\pi^{-1}$. Also each trajectory of the flow defined by $dH^{\#}$ which lies in N is mapped onto a trajectory of $dh^{\#}$ by the map π. Also π preserves parametrization of the trajectories.

Proof: Use the same notation and coordinate system as in the proof of the previous theorem. By hypothesis $0 = \{H, f\} = \Omega(dH^{\#}, df^{\#})$ for all $f \in \mathfrak{A}$ and so $\Omega(dH^{\#}, S^{\#}) = 0$. The vector field $dG^{\#}$ constructed in the previous section has the property that $dG^{\#}(s) \in S^{\#}_s$ for all $s \in N$. Thus $\Omega_s(dH^{\#}(s), dG^{\#}(s)) = 0$ for all $s \in N$ or H is an integral for the flow defined by $dG^{\#}$ restricted to N. Since H is constant along the trajectories of $dG^{\#}$ in N we have $H(m) = H(r)$. But r was an arbitrary point of L_m near m and L_m is connected. Thus H is constant on L_m.

In order to establish the rest of the theorem it is enough to show that $D\pi(dH^{\#}(m)) = dh^{\#}(m)$ when $\pi(m) = b$. But this follows at once from $\pi^* h = H|N$ or $\pi^* dh = d(H|N)$ and $\pi^* \omega = \Omega|N$.

3. *Miscellaneous Remarks and Applications.*

a) Symmetries: In [3], [4] and [6] Hamiltonian systems which are invariant under the action of a Lie group are studied and a reduction which motivated the results of the previous section is given. Here we shall show how the results given above apply to this case. Let G be a connected Lie group, A its algebra and $\psi_a : G \times M \to M : (g, m) \to \psi(g,m) = gm$ an action of G on M such that $\psi(g,\cdot) : M \to M$ is a symplectic diffeomorphism for all $g \in G$. Let $a \in A$ and e^{at} be the one parameter subgroup of G generated by a. Then

$$\psi_a : R \times M \to M : (t, m) \to \psi(e^{at}, m)$$

is a Hamiltonian flow on M and so is generated by a local Hamiltonian vector field X_a on M. X_a is a local Hamiltonian vector field in that $X_a^\flat$ is a closed one form. Let us assume that for each $a \in A$ the form $X_a^\flat$ is exact i.e. X_a is a global Hamiltonian vector field. (This is always the case if Ω is exact, see [3]). Then $X_a = dF_a^\#$ where $F_a : M \to R$ is a function which is determined up to an additive constant. The map which associates to each $a \in A$ the vector field X_a is a Lie algebra homomorphism from A into $\mathfrak{X}$. In general it is not possible to choose the additive constants so that there is a Lie algebra homomorphism from A into $\mathfrak{F}$ which takes a into F_a where $X_a = dF_a^\#$. This problem is discussed in detail in [8] but will not concern us here. Let $\mathfrak{A} \subset \mathfrak{F}$ be the set of all functions $F_a : M \to R$ such that $X_a = dF_a^\#$ generates a flow $\psi(e^{at}, m) = e^{at} m$ for some $a \in A$. Even though $\mathfrak{A}$ may not be a homomorphic image of A, but it is a finite dimensional subalgebra of $\mathfrak{F}$.

Let $m \in M$ be fixed. Then $\psi(-, m) : G \to M$ and $D_1\psi(e,m) : A = T_eG \to T_mM$. Since

$$X_a(m) = dF_a^\#(m) = \frac{d}{dt}\psi(e^{at}, m)\Big|_{t=0} = D_1\psi(e,m)(a)$$

we see that $S^{\#}_m = D_1\psi(e,m)(A)$. Let us assume, as in the previous section that there exists an integral manifold N through m for S^0. Let $G_N = \{ g \varepsilon G : gN = N \}$. Clearly G_N is a closed subgroup of G and hence a Lie subgroup. Let $A_N \subset A$ be the algebra of G_N. Let $f \varepsilon \mathfrak{A}$ and since f is constant on N it follows that $f(e^{at} m) = f(\psi(e^{at}, m))$ is constant for each $a \varepsilon A_N$. Thus

$$0 = \frac{d}{dt} f(\psi(e^{at},m))\Big|_{t=0} = df(D_1\psi(e,m)(a))$$

or

$$D_1\psi(e,m)(A_N) \subset S^0_m.$$

Since $D_1\psi(e,m)(A_N) \subset D_1\psi(e,m)(A) = S^{\#}_m$ we can combine these results to give

$$D_1\psi(e,m)(A_N) \subset S^0_m \cap S^{\#}_m .$$

To show the inclusion in the opposite direction let $g \varepsilon \mathfrak{A}$ be any element such that $dg(m)^{\#} \varepsilon S^0_m \cap S^{\#}_m$. Let $a \varepsilon A$ be such that $\psi(e^{at}, m)$ is the flow generated by $dg^{\#}$. The opposite inclusion will follow once it is shown that $a \varepsilon A_N$ or that the flow $\psi(e^{at},m)$ leaves N invariant. This follows from $\{f,g\}(r) = 0$ for all $f \varepsilon \mathfrak{A}$ and all $r \varepsilon N$ which in turn follows from $d\{f,g\} = 0$ on N and $\{f,g\}(m) = 0$. But $\{f,g\}(m) = 0$ since $dg(m)^{\#} \varepsilon S^0_m \cap S^{\#}_m$ and $d\{f,g\} = 0$ on N since $\{f,g\} \varepsilon \mathfrak{A}$ and the elements of $\mathfrak{A}$ are constant on N. Thus combining the above

$$D_1\psi(e,m)(A_N) = S^0_m \cap S^{\#}_m .$$

Thus the integral manifold of $S^0 \cap S^{\#}$ on N through m is just the orbit of m under the action of G_N on N. Thus the results of the previous section are natural generalizations of the results of [3], [4] and [6].

b) Local Coordinates : If $\mathfrak{A} \subset \mathfrak{F}$ is a finite dimensional subalgebra then a classical theorem of Lie [2] can be used in many cases to choose local coordinates on M so that the functions in $\mathfrak{A}$ have a simple form.

A k-tuple $(f_1, \ldots, f_k)$ of functions on a symplectic manifold (M, Ω) is called complete if the differential $df_1, \ldots, df_k$ are independent and if there exist functions $U_{ij} : R^k \to R$ $(1 \leq i, j \leq k)$ such that

$$\{f_i, f_j\} = U_{ij}(f_i, \ldots, f_k) \quad \text{for} \quad 1 \leq i, j \leq k .$$

The matrix (U_{ij}) of functions is called the structural matrix.

LIE'S THEOREM: Let $(f_1, \ldots, f_k)$ and $(f'_1, \ldots, f'_k)$ be complete k-tuples on the symplectic manifolds (M, Ω) and (M', Ω') respectively and $\dim M = \dim M'$. Suppose that $f_i(x) = f'_i(x')$ for some $x \in M$ and $x' \in M'$. Then there exists a diffeomorphism ϕ from a neighborhood of x onto a neighborhood of x' such that $\phi^*\Omega' = \Omega, \phi^* f'_i = f_i$ if and only if the two k-tuples have the same structural matrix.

In the special case when the U_{ij} are linear more information can be obtained. Assume that

$$\{f_i, f_j\} = C^h_{ij} f_h$$

where C^h_{ij} are constants, the structural constants. (In this subsection the usual conventions of tensor analysis are employed). Then since $\{ , \}$ is skew symmetric $C^h_{ij} + C^h_{ij} = 0$ and Jacobi's identify for $\{ , \}$ gives

$$C^\alpha_{\beta i} C^\beta_{jk} + C^\alpha_{\beta j} C^\beta_{ki} + C^\alpha_{\beta k} C^\beta_{ij} = 0 .$$

Let $q^1, \ldots q^n$ $p_1, \ldots p_n$ be the usual symplectic coordinates in R^{2n} and define

$$F_i = C^{\alpha}_{i\beta}\, q^{\beta}\, p_{\alpha}\,.$$

Then

$$\{F_i, F_j\} = \{ C^{\alpha}_{i\beta}\, q^{\beta} p_{\alpha},\ C^{a}_{jb}\, q^{b} p_{a} \}$$

$$= C^{\alpha}_{i\beta}\, C^{a}_{jb} \{ q^{\beta} p_{\alpha},\ q^{b} p_{a} \}$$

$$= C^{\alpha}_{i\beta}\, C^{a}_{jb} [\, \delta_{a\beta}\, p_{\alpha} q^{b} - \delta_{\alpha b}\, q^{\beta} p_{a} \,]$$

$$= [\, - C^{m}_{\beta i}\, C^{\beta}_{jn} - C^{m}_{\beta j}\, C^{\beta}_{ni} \,]\, p_{m} q^{n}$$

$$= C^{\beta}_{ij}\, C^{m}_{\beta n}\, p_{m} q^{n}$$

$$= C^{\beta}_{ij}\, F_{\beta}\,.$$

Thus the k-tuples $(f_1, \ldots, f_k)$ and $(F_1, \ldots, F_k)$ have the same structural constants. Thus if $x \in M$ is such that $df_1(x), \ldots, df_k(x)$ are independent and $x' \in R^{2n}$ is such that $f_i(x) = F_i(x')$ and $dF_1(x'), \ldots, dF_k(x')$ are independent there exists a symplectic coordinate system q, p about x such that in these coördinates $f_i(q,p) = F_i(q,p)$.

For example if $f_1, f_2, f_3 \in \mathcal{F}$ are such that $\{f_i, f_j\} = f_k$ when (i, j, k) is an even permutation of $(1, 2, 3)$ and $\{f_i, f_j\} = -f_k$ when (i, j, k) is an odd permutation of $(1, 2, 3)$ then the corresponding functions F_1, F_2, F_3 are

$$F_1 = q^2 p_3 - q^3 p_2$$

$$F_2 = q^3 p_1 - q^1 p_3$$

$$F_3 = q^1 p_2 - q^2 p_1\,.$$

These are the usual forms of angular momentum in R^6 .

c) Holonomic Systems : Here we shall use the general results of section 2 to give a derivation of Hamilton's equations of motion for a holonomic system. The derivation is unique in that it proceeds directly from the Newtonian formulation to the Hamiltonian formulation without the historical intermediate Lagrangian formulation.

Let $F = (f_1, \ldots, f_q) : R^n \to R^q$, $1 \leq q \leq n-1$, be a smooth function such that $0 \in R^q$ is a regular value and so $P = F^{-1}(0)$ is a regularly embedded submanifold of R^n of dimension $l = n-q$. In the physical system the function F represents the constraints and P the configuration space.

Since TR^n is naturally diffeomorphic to $R^n \times R^n$ we shall use coordinates $(x,y) \in R^n \times R^n$ where x is considered as a coordinate in the position space R^n and y is considered as a coordinate in the velocity space $T_x R^n$. Let G be a positive definite symmetric matrix and $K = (1/2)\,y^T G y = (1/2) g_{\alpha\beta} y^\alpha y^\beta$. Consider K as a Riemannian metric on R^n and the kinetic energy of the physical system. We shall take G as constant i.e. independent of x in order to simplify the calculations given below. Define a symplectic structure on TR^n by

$$\Omega = dx^T \wedge d(Gy) = g_{\alpha\beta}\, dx^\alpha \wedge dy^\beta \; .$$

In the natural way consider $F_1, \ldots F_q$ as functions on TR^n and since they are independent of y they are involutions . That is $\{ F_i, F_j \} = 0$ for all i and j where $\{ , \}$ is the Poisson bracket operator defined by $\cdot\Omega$. Let $O \subset R^n$ be an open neighborhood of P such that $dF_1, \ldots, dF_q$ are independent at all points of O and $M = O \times R^n$. Let $\mathfrak{A}$ be the algebra generated by $F_1, \ldots, F_q$. By the definition of M it is clear that $\mathcal{A}$ has rank q.

In summary we have used the constraint $F_1, \ldots, F_q$ and the kinetic energy K to define a symplectic manifold (M, Ω) and an algebra of functions $\mathcal{A}$ of rank q. This is precisely the data necessary for the theorems of section 2.

LEMMA : There is a naturally defined symplectic diffeomorphism between the quotient space of Theorem 1 and the cotangent bundle of P.

Proof: We use $\mathcal{A}$ to define $S, S^{\#}$ etc. as in the previous section. It is clear that $P \times R^n = \bigcup_{s \in P} T_s R^n$ is an integral manifold of S^0 and let it be called N.

Since $0 = \{F_i, F_j\} = (dF_j^{\#}) \rfloor dF_i$ then $S_m^{\#} \subset S_m^0$. Let $m \in N$ have coordinates (u, v). A direct calculation yields

$$dF_i(m) = \frac{\partial F_i}{\partial x^j}(u)\, dx^j \quad \text{and} \quad dF_i(m)^{\#} = g^{\alpha\beta} \frac{\partial F_i}{\partial x^{\alpha}}(u) \frac{\partial}{\partial y^{\beta}}$$

where $G^{-1} = \{g^{\alpha\beta}\}$. Thus the spaces S_m, S_m^0 and $S_m^{\#}$ are independent of v and so we may consider $S_m^{\#}$ as a subspace of $T_u R^n$ and $S_m^{\#} = L_m$. Thus $B = N/\sim = \bigcup_{p \in P} (T_p R^n \mid Q_p)$ where $\dot{y} \in Q_p \subset R_n$ if $y = G^{-1} w$ where $w^T z = 0$ for all $z \in T_p P$.

Now construct the map $\phi : B \to T^* P$ by sending $[y] \to f_y$ where $f_y(z) = y^T G z$. First ϕ is well defined since $[y] = [y + k]$ when $k = G^{-1} w$ where $w^T z = 0$ for all $z \in T_p P$ and $f_y + k(z) = y^T G z + k^T G z = y^T G z + w^T G^{-1} G z = y^T G z = f_y(z)$ when $z \in T_p P$. The map is clearly one-to-one and since the dimensions of the two vector spaces are the same, ϕ is an isomorphism. Thus

$\phi : B \to T^* P$ is a diffeomorphism. Also it is clear that ϕ takes the simplectic structure of B to the natural symplectic structure of $T^* P$ and so ϕ is a symplectic diffeomorphism.

Now let us consider a system of particles whose position and velocity are given by $(x, y) \in R^n \times R^n = TR^n$ and let the system be subjected to "ideal forces" which constraint it to move on $P = F^{-1}(0)$. Let the kinetic energy of the system be $K = (1/2)\, y^T G_y$ and the external forces be derived from the potential energy U. The assumption that the forces of constraint are "ideal" means in this notation that the equations of motion for the system is of the form $h^\# = dH^\# + \theta^\#$ where $H = K + U$, θ is a smooth one form such that $\theta(m) \in S_m$ for all m and $\#$ is with respect to the symplectic structure defined above. Since the motion is to take place on P you must have $h(m)^\# \in S^0_m$ for all m and so $h^\# \rfloor dF^\#_i = h^\# \rfloor dF^\#_i \rfloor \Omega = 0$ or $\{ H, F_i \} = 0$ on N. Thus the previous theory applies, i.e. Theorem 2, and so the flow defined by $h^\#$ may be carried down to B and across by ϕ to $T^* P$. This gives rise to Hamilton's equations of motion on T^*P.

d) Characteristic Multipliers of a Periodic Solution : In [4] Meyer gave a generalization of an inequality by Poincare on the algebraic multiplicity of the characteristic multiplier $+1$ of a periodic solution. The hypotheses given there were far too strong and the proof yields a better theorem. Since the statement of the improvided theorem uses the notation of this paper we shall give it here but refer the reader to [4] for the proof.

Using the notation of section 2, $\mathfrak{A}$ is an algebra of integrals for the Hamiltonian vector field $X = (dH)^\#$, $S_m = \{ dF(m) : F \in \mathfrak{A} \}$, and S^0_m and $S^\#_m$ are as before. In constrast to the previous results it is best to include H as an element of $\mathcal{Q}$. Let the solution of X through m be periodic of least non negative period T. (Note we include the case of an equilibrium point).

THEOREM 3 : The geometric multiplicity of the characteristic multiplier $+1$ *of the periodic solution through* m *is greater than or equal to* $\dim S_m$. *The algebraic multiplicity of the characteristic multiplier* $+1$ *of the periodic solution through* m *is greater than or equal to* $\dim S_m + \dim(S_m^0 \cap S_m^\#)$.

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