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A NOTE ON UNIVERSAL MAPS

by

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ABSTRACT

A map d of the n-dimension Euclidean unit ball B^n into itself is called universal if every map of B^n into itself agrees with d at at least one point. *Theorem.* Let d be a map of B^n into itself, let $A = d^{-1}(S^{n-1})$ where S^{n-1} is the boundary of B^n, and let f be the restriction of d to A. Then d is universal if and only if the homomorphism generated by f between the corresponding Čech cohomology groups $f^*: H^{n-1}(S^{n-1}) \to H^{n-1}(A)$ is nontrivial.

A map f from a topological space X into a topological space Y is called universal [1] if every map from X into Y has a coincidence with f, that is for each map g from X into Y there is an $x \in X$ such that $f(x) = g(x)$. In [3] a sufficient condition that a map from the Euclidean n-ball B^n into itself be universal was established.

THEOREM 1. (Schirmer). If f is a self mapping of B^n that maps the boun-

dary of B^n , S^{n-1}, *onto itself essentially, then* f *is universal.*

In this note a necessary and sufficient condition that a map be universal is established. The author is indebted to Professor Chung-Wu Ho for raising the question of the existence of necessary and sufficient conditions for a self mapping of the 2-ball be universal. The author is also indebted to Professor M. Dold for pointing out Hopf's extension theorem [2] during a conversation, which eventually eliminated a long proof of the two dimensional case and resulted in the completion of the proof of the n-dimensional case.

In what follows $H^n(X)$ shall denote the nth Čech cohomology group of a space X, with integer coefficients.

THEOREM 2. *Let* f *be a mapping of a space* X *into* B^{n+1}, $A = f^{-1}(S^n)$, *and* $g : A \to S^n$ *such that* $g(x) = f(x)$ *for* $x \in A$. *Then* f *is not universal if and only if the map* g *extends to a map* $G : X \to S^n$

Proof. If f is not universal there is a map h of X into B^{n+1} such that $f(x) \neq h(x)$ for x in X. Let $G(x)$ be the point of intersection of S^n and the open ray that contains $f(x)$ and has $h(x)$ as an endpoint. Clearly $G(x) = g(x)$ for x in A. On the other hand, if there is an extension of g to a map $G : X \to S^n$ let $h(x) = -G(x)$ for x in X. If $x \notin A$ then $h(x) \in S^n$ and $f(x) \notin S^n$ so $h(x) \neq f(x)$; if $x \in A$ then $h(x) = -G(x) = -g(x) = -f(x) \neq f(x)$. Therefore, f is not universal.

THEOREM 3. *Hopf's extension theorem. Let* X *be a compact metric space of dimension* $\leq n+1$, h *a mapping of a closed subset* A *of* X *into* S^n, *and* e *a generator of* $H^n(S^n)$. *Then in order that* h *be extendable over* X *it is* *necessary and sufficient that* $h^*(e)$ *be extendable over* X, *where* $h^* : H^n(S^n) \to H^n(A)$ *is the homomorphism induced by* h.

THEOREM 4. Let f, A, *and* g *be as in Theorem 2. If* X *is separable metric and the dimension of* X *is* $\leq n+1$ *and* $H^n(X) = 0$ *then* f *is not universal if and only if the induced homomorphism* $g^* : H^n(S^n) \to H^n(A)$ *is the zero homomorphism.*

Proof. Suppose there exists a $G : X \to S^n$ that extends g over X. Then $g = G \circ j$ where $j : A \to X$ is the inclusion map. Since by hypothesis $H^n(X) = 0$, $G^* = 0$ and $g^* = j^* \circ G^* = 0$. If $g^* = 0$, then in particular, $g^*(e) = 0$ for any generator e of $H^n(S^n)$ and it follows that $g^*(e)$ can be trivially extended to X. Hopf's extension theorem then asserts that g can be extended over X, by Theorem 2 f is not universal.

THEOREM 5. Let f, A, *and* g *be as in Theorem 2 and let* $X = B^{n+1}$. *Then* f *is universal if and only if* g *is essential.*

Proof. If g is not essential then g is homotopic to a constant map c. Therefore, $g^* = c^* = 0$ and by Hopf's extension theorem g can be extended to a map $G : X \to S^n$. f is not universal by Theorem 2. If on the other hand, g is essential, then since all mappings of B^{n+1} into S^n are homotopic to a constant map it follows that g cannot be extended over B^{n+1}. Therefore, f is universal by Theorem 2.

Remarks. (i) The condition $X = B^{n+1}$ in Theorem 5 could have been replaced by the condition "X is compact metric and every map of X into S^n is not essential".

(ii) The use of Čech cohomology could have been replaced by Čech homology but not by singular homology. For example, let $X = B^2$ and let A be the boundary of a neighborhood of the origin that is contained in $\{ z : 1/2 \leq |z| \leq 1 \}$ and has trivial singular homology groups. Let $d(z, a)$ be the distance from z to A. Let $f : B^2 \to B^2$ be defined by $f(z) = (2 - d(z, A))\, z$ if $|z| \leq 1/2$, and $f(z) = (1/|z| - d(z, A))\, z$ if $|z| \geq 1/2$. Then f is in fact universal and $f^{-1}(S^1) = A$.

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