Revista Colombiana de Matemáticas
Volumen IX (1975), págs. 115 - 124

ON SOME DISTRIBUTIONS INVOLVING GENERAL FUNCTIONS

by

G. S. LINGAPPAIAH

SUMMARY

In this note, two aspects in relation to distribution involving general functions are considered. Firstly, the distribution of the sum and also the ratio of two independent radom variables *with* densities including $\phi_2(a_1,\dots a_n; S; \lambda_1 x, \dots, \lambda_n x)$ and ψ_2 $(p; b_1 \dots, b_n; t_1 u \dots t_n y)$ is dealt with. The special case of $n=2$ is considered. Also, the distribution of the sum of variables with ϕ_2 and ϕ_3 as their densities is obtained. The second aspect is estimation. In ϕ_2, a parameter is estimated and the estimate is put in the closed form in terms of the general function ψ_2. Also, the Bayes estimate in the distribution involving the Bessel function $I_\rho(cx)$ is obtained and is expressed in terms of another general function F_A.

1. *Introduction.* General functions are recently being used often in statistical methods both to describe the random phenomenon as well as to discuss the properties of such phenomenon. Some of the main reasons for this being the compact forms

in which, heavy distributions can be expressed in as much as the ease with which the properties of these distributions can be expressed in terms of general functions , utilizing the various properties of these functions. For example, in [1] and [3] we find such general functions as ψ_2, ϕ_2, Bessel functions, and confluent hypergeometric functions are used to describe the random variables in a radar system. In [2] some of the functions are applied to weapon system analysis. In [4] one finds the generalizations of the results of [3]. In [8] Braaksma's H-function is utilized in connection with the testing and the estimation of the normal covariance matrix with intra-class structure. In [9], [10] the distributions of the product on independent Gamma and Beta variables are expressed in terms of Meijer's G-function. Non-central chi-square as well as the non-central-F distributions are discussed in terms of general functions in [11]. In [12! the Bayes' estimates of the non-centrality parameter in the multi-variate analysis are put in compact forms using general functions. In this paper, such general distributions are treated under two aspects. Distribution of the sum and quotient of two random variables from these general distributions is considered first. Secondly, the estimation problem is taken on.

2. *Distribution of* $x+y$

2(i) : Let the independent variables x and y have their corresponding densities as

$$f(x) = (\alpha-\lambda_1)^{a_1}(\alpha-\lambda_2)^{a_2} e^{-\alpha x} x^{S-1} \alpha^{S} \cdot \phi_2(a_1, a_2 ; S ; \lambda_1 x, \lambda_2 x)$$

$$\div \Gamma(S)\, \alpha^{a_1+a_2}, \quad x > 0 \tag{1}$$

$$f(y) = e^{-r^2/2\lambda} (2\lambda\alpha t)^{Q} \alpha^{P} e^{-\alpha t y} y^{P-1} \psi_2(Q ; Q, P : r^2 \alpha t, \alpha t y) \div \Gamma(P) \tag{2}$$

$$\alpha, \lambda, y > 0, \quad r \geq 0, \quad t = 1/(1-2\lambda\alpha)$$

where $\phi_2 \cdot \psi_2$ are the general functions [5].

Then, the joint density can be written as

$$K \cdot M \cdot \sum_1 \sum_2 \frac{(a_1)_{\delta_1} (a_2)_{\delta_2}}{} \frac{(Q)_\theta}{(Q)_{\theta_1} (P)_{\theta_2}} e^{-\alpha(x+y)} x^{\delta} y^{\theta_2}$$

$$\cdot \frac{(\lambda_1)^{\delta_1}}{\delta_1 !} \frac{(\lambda_2)^{\delta_2}}{\delta_2 !} \frac{(r^2 \alpha t)^{\theta_1}}{\theta_1 !} \frac{(\alpha t)^{\theta_2}}{\theta_2 !} \tag{3}$$

where $M = \alpha^{S+P} \Gamma^{-1}(S) \Gamma^{-1}(P)$

$$K = e^{-r^2/2\lambda} (2\lambda\alpha t)^Q \left(1 - \frac{\lambda_1}{\alpha}\right)^{a_1} \left(1 - \frac{\lambda_2}{\alpha}\right)^{a_2},$$

Σ_1, Σ_2 are on δ_1, δ_2 and θ_1, θ_2 respectively, all from 0 to ∞. From (3), we get the distribution of the sum $x + y = u$ as

$$f(u) = K \cdot g(u; \alpha, p + S).$$

$$\sum_{j=o}^{\infty} \frac{(Q)_j (\alpha t u)^j}{(p+S)_j \, j!} \; {}_1F_1(Q+j; Q; r^2 \alpha t) \, \phi_2(a_1, a_2; p+S+j; \lambda_1 u, \lambda_2 u) \tag{4}$$

where $g(u; \alpha; p)$ is the Gamma density with the parameters α, P and ${}_1F_1$ is the confluent hyper-geometric function [5].

It is trivial to show that (4) is indeed a density function by using [5] and [6].

2(ii) : The distribution function of (4) is

$$F(u) = K \cdot \sum_{i,j,\ell} \frac{(a_1)_i (a_2)_j (Q)_\ell}{i! \, j! \, \ell!} \left(\frac{\lambda_1}{\alpha}\right)^i \left(\frac{\lambda_2}{\alpha}\right)^j (t)^\ell$$

$$\cdot {}_1F_1(Q+\ell; Q; r^2 \alpha t) \, G(u; p+S+\ell i+j; \alpha) \tag{4a}$$

where $G(u;p;\alpha)$ is the Gamma probability integral.

2 (iii) : The corresponding characteristic function is

$$\phi(z) = e^{-r^2/2\lambda}(2\lambda\alpha t)^{Q} \sum_{j=o} [\alpha/(\alpha-iz)]^{p+S+j} \frac{(Q)_j\, t^j}{j!}$$

$$\cdot {}_1F_1(Q+j\,;\,Q\,;r^2\alpha\, t)\cdot m_1^{a_1}\; m_2^{a_2} \tag{5}$$

where $$m_1 = \left[\left(1-\frac{\lambda_1}{\alpha}\right)\Big/\left(1-\frac{\lambda_1}{\alpha-iz}\right)\right]^{a_1}$$

and $$m_2 = \left[\left(1-\frac{\lambda_2}{\alpha}\right)\Big/\left(1-\frac{\lambda_2}{\alpha-iz}\right)\right]^{a_2}$$

(5) can be written as

$$[\,2\lambda\alpha(\alpha-iz)/T\,]^{Q}\, m_1^{a_1}\, m_2^{a_2}\,[\,\alpha/(\alpha-iz)\,]^{p+S}\, e^{-izr^2/2\lambda T} \tag{6}$$

where $T = 2\lambda\alpha^2 - iz(1+2\lambda\alpha)$.

2 (iv) : The generalization follows immediately.

If $$f(x) = [\prod_{i=1}^{n}(1-\frac{\lambda_i}{\alpha})^{a_i}]^{-1}\,\Gamma(S)\, e^{-\alpha x}\cdot\alpha^{S}\, x^{S-1}$$

$$\cdot\,\phi_2(a_1,\ldots,a_n\,;\,S\,;\lambda_1 x,\ldots\lambda_n x)\,,\quad x>0\,, \tag{7}$$

and $$f(y) = [\,F_c(\beta,p)]^{-1}\, e^{-\alpha y}\,\alpha^{p}\, y^{p-1}\,\Gamma^{-1}(p)\cdot\psi_2[\beta\,;b_1,\ldots b_n;t_1y\ldots t_ny]\,.\ y>0 \tag{8}$$

where $F_c(\beta,p)$ is the polynomial in [4], and ψ_2, ϕ_2 are are general functions.

P. 445 [7] that is,

$$F_c(\beta\,;p) = \sum \frac{(\beta)_\theta\ (P)_\theta}{(b_1)_{\theta_1}\ldots(b_n)_{\theta_n}} \left(\frac{t_1}{\alpha}\right)\ldots\left(\frac{t_n}{\alpha}\right) \tag{9}$$

where $\theta = \sum_i \theta_i$ and Σ on $\theta_i^{\prime}S$ from 0 to ∞ .

The joint density now is,

$$\prod_{i=1}^{n}\left(1-\frac{\lambda_i}{\alpha}\right)^{a_i} e^{-\alpha(x+y)}\,\alpha^{p+S}\,\Gamma^{-1}(S)\,\Gamma^{-1}(p)\,.\,[F_c(\beta,p)]^{-1}$$

$$\cdot\,\Sigma_1\,\Sigma_2\left[\prod_{i=1}^{n}\frac{(a_i)_{\delta_i}}{\delta_i\,!}\;\frac{(t_i\quad)^{\theta_i}}{\theta_i\,!\,(b_i)_{\theta_i}}\right]\frac{(\beta)_\theta}{(S)_\delta}\,x^{\delta+S-1}\,y^{\theta+p-1} \tag{10}$$

where $\delta = \sum_i \delta_i$ and Σ_1, Σ_2 on $\delta_1\ldots\delta_n$ and $\theta_1\ldots\theta_n$, respectively , all from 0 to ∞ .

From (10), we get the distribution of $u = x+y$ as

$$\sum_2 \frac{(\beta)_\theta\,(p)_\theta}{(p+S)_\theta}\,g(u\,;\alpha\,;p+S)\cdot\phi_2\,[a_1,\ldots,a_n\,;\,p+S+\theta\,;\,\lambda_1 u,\ldots,\lambda_n u]$$

$$\cdot\,[F_c(\beta\,.\,p)]^{-1}\left[\prod_{i=1}^{n}\left(1-\frac{\lambda_i}{\alpha}\right)^{a_i}\frac{(ut_i/\alpha)^{\theta_i}}{\theta_i!\,(b_i)_{\theta_i}}\right]. \tag{11}$$

2 (v) : Now suppose y , instead of (2) , has the density of the form

$$e^{-c/\alpha}\left(1-\frac{b}{\alpha}\right)^{\beta} e^{-\alpha y}\,\alpha^p\,y^{p-1}\,\Gamma^{-1}(\alpha)\,\phi_3(\beta\,;p\,;by,cy)\ ,\quad y>0 \tag{12}$$

Then, developing on the same lines as (3), we have the distribution of the sum ,

$$f(u) = \left(1-\frac{\lambda_1}{\alpha}\right)^{a_1}\left(1-\frac{\lambda_2}{\alpha}\right)^{a_2}\left(1-\frac{b}{\alpha}\right)^{\beta} e^{-c/\alpha}\cdot\sum_{j=0}^{\infty} g(u\,;\alpha\,;p+S+j)\cdot[(c/\alpha)^j/j!]$$

$$\cdot \Phi_2 [a_1, a_2, \beta ; p+S+j ; \lambda_1 u, \lambda_2 u, bu] \quad , \quad u > 0 \tag{13}$$

3. *Distribution of* x/y

3(*i*) : Now considering again (1) and (2), and making the transformation $x/y = \nu$, we have from (3)

$$f(\zeta) = K \cdot B(P, S ; \zeta) \sum_{j=o}^{\infty} F_1 \left[P+S+j ; a_1, a_2, S ; \frac{\lambda_1}{\alpha} \zeta , \frac{\lambda_2}{\alpha} \zeta \right]$$

$$\cdot {}_1F_1 (Q+j ; Q ; r^2 \alpha t) \frac{(P+S)_j \, (Q)_j}{(P)_j \; j!} [t(1-\zeta)]^j$$

$$, \; 0 < \zeta < 1 \; , \tag{14}$$

where $\zeta = \nu/(1+\nu)$ and F_1 is the general function as in [5], $B(P ; S ; \zeta)$ is the complete Beta function with parameters P, S.

3(*ii*) : In general, if the variables, x, y have the distributions (7) and (8), then by using (9) we get the density of the ratio $x/y = \nu$ as

$$\prod_{i=1}^{n} \left(1 - \frac{\lambda_i}{\alpha} \right)^{a_i} B(P, S : \zeta) [F_c (\beta, P)]^{-1}$$

$$\cdot \Sigma_1 \frac{(p+S)\,\delta}{(S)_\delta} F_c \left[\beta ; p+S+\delta ; b_1, \ldots b_n ; \frac{t_1}{\alpha} (1-\zeta), \ldots, \frac{t_n (1-\zeta)}{\alpha} \right]$$

$$\cdot \prod_{i=1}^{n} \left[\frac{(a_i)_{\delta_i}}{\delta_i !} \frac{(\lambda_i \zeta / \alpha)^{\delta_i}}{\delta_i !} \right] \tag{15}$$

4. *Estimation*

4 (*i*) : Now consider (12) and take a sample of size n from the population,

then (for known α, β) the likelihood function is ,

$$L(x/c) = \left(1 - \frac{b}{\alpha}\right)^{n\beta} e^{-nc/\alpha} e^{-\alpha n\bar{x}} \left(\prod_{i=1}^{n} x_i^{p-1}\right) \Gamma^{-n}(P)$$

$$\cdot \Sigma_1 \Sigma_2 \prod_{i=1}^{n} \left[\frac{(\beta)_{a_i}}{(P)\, a_i + b_i} \; \frac{(bx_i)^{a_i}}{a_i\,!} \; \frac{(cx_i)^{b_i}}{b_i\,!} \right] \tag{16}$$

where $x = x_1, \ldots, x_n$; and Σ_1, Σ_2 on $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. Now if c has the prior $f(c) = e^{-c} c^{m-1} / \Gamma(m)$, then,

$$L(c/x) = \frac{t^m \Sigma_1 \Sigma_2 \prod_{i=1}^{n} \left[\frac{(\beta) a_i}{(P)_{a_i + b_i}} \; \frac{(bx_i)^{a_i}}{a_i!} \; \frac{(x_i)^{b_i}}{b_i!} \right] c^{B + m - 1} e^{-ct}}{\Sigma_1 \prod_{i=1}^{n} \left[\frac{(\beta)\, a_i}{(P)_{b_i}} \; \frac{(bx_i)^{a_i}}{a_i!} \right] \psi_2 \left[m ; p + a_1, \ldots p + a_n ; t^{-1} x_1, \ldots t^{-1} x_n \right]}$$

$$, \; B = \sum_i b_i \; . \tag{17}$$

where $t = 1 + (n/\alpha)$ and ψ_2 is the general function [7] then, we have the estimate of c as

$$E(c/x) = \frac{m}{t} \; \frac{\Sigma_1 \, \phi(\beta, P, x)\, \psi_2(m+1)}{\Sigma_1 \phi(\beta, P, x)\, \psi_2(m)} \tag{18}$$

$$\text{where} \quad \phi(\beta, P, x) = \prod_{i=1}^{n} \left[\frac{(\beta)\, a_i}{(P)\, a_i} \; \frac{(bx_i)^{a_i}}{a_i!} \right]$$

$$\psi_2(m) = \psi_2 \left[m ; p + a_1, \ldots, p + a_n ; t^{-1} x_1, \ldots t^{-1} x_n \right]$$

4 *(ii)* : Consider the distribution

$$(p/q)^{r/2}\,(r/x)\,e^{-cx}\,I_r(2cx\sqrt{pq}) \tag{19}$$

$r, c, x > 0$, $p+q=1$ and $I_r(x)$ is the Bessel function. (19) can be written as

$$(p/q)^{r/2} e^{-cx} (r/x) \left[\frac{(xc\sqrt{pq})^r}{\Gamma(r+1)} e^{-2c\sqrt{pq}\,x} \phi\left(r+\frac{1}{2}\,;2r+1\,;4cx\sqrt{pq}\right) \right] \tag{20}$$

where ϕ is the confluent hypergeometric function. It is easy to show that (19) and (20) are indeed densities by using [5] and [6] and carefully replacing 1 by $p+q$. If we have a sample of size n , we have from (20), the likelihood function (for given r and p)

$$L(x/c) = (pc)^{nr} e^{-[cn\bar{x} + 2c\,n\bar{x}\sqrt{pq}]} r^n \Gamma^{-n}(r+1) \left(\prod_{i=1}^{n} x_i^{r-1} \right)$$

$$\cdot \sum \left[\prod_{i=1}^{n} \frac{(r+\frac{1}{2})_{a_i}}{(2r+1)_{a_i}} \frac{(4c\sqrt{pq})^{a_i}}{a_i!} x_i^{a_i} \right] \tag{21}$$

where Σ runs over $a_1, \ldots, a_n$ from 0 to ∞ and $x = x_1, \ldots, x_n$.

Now if c has the prior $f(c) = e^{-c}$, $c > 0$, then from (21) we get ,

$$L(c/x) = \frac{z^{nr+1}}{\Gamma(nr+1)} \frac{\sum e^{-cz} c^{nr+A} h(r,x)}{F_A\left[nr+1\,;r+\frac{1}{2}, \ldots, r+\frac{1}{2}\,;2r+1, \ldots, 2r+1\,; \frac{tx_1}{z}, \frac{tx_2}{z}, \ldots, \frac{tx_n}{z} \right]} \tag{22}$$

where $A = \Sigma\, a_i$ and

$$h(r,x) = \prod_{i=1}^{n} \left[\frac{(r+\frac{1}{2})_{a_i}}{(2r+1)_{a_i}} \frac{(4c\sqrt{pq})^{a_i}}{a_i!} x_i^{a_i} \right]$$

and $z = 1 + n\bar{x}(1 + 2\sqrt{pq})$, $t = 4\sqrt{pq}$.

From (22) we have, the estimate of c as

$$E(c/x) = \frac{nr+1}{z} \frac{F_A(nr+2)}{F_A(nr+1)} \tag{23}$$

where $F_A(nr+1)$ is as in (22) and $F_A(nr+2)$ accordingly. F_A is the hypergeometric function p. 445 [7] .

If $p=q$, and $r=1$, we have

$$E(c/x) = \frac{n+1}{1+2n\bar{x}} \frac{F_A\left[n+2\,;\,\frac{3}{2},\ldots\frac{3}{2}\,;\,3,\ldots\frac{2x_1}{1+2n\bar{x}},\ldots\frac{2x_n}{1+2n\bar{x}}\right]}{F_A\left[n+1\,;\,\frac{3}{2},\ldots\frac{3}{2}\,;\,3,\ldots\frac{2x_1}{1+2n\bar{x}},\ldots\frac{2x_n}{1+2n\bar{x}}\right]} . \tag{24}$$

References

1. *McNolty Frank (1967). Applications of Bessel Function Distributions.* Sankhya-Series - B Vol. 29, p. 235-248.

2. ______________ *(1962) Kill Probability When the Weapon Bias is Randomly Distributed.* Operations Research Vol. 10, p. 693-701.

3. *McNolty, F. and Timsky. J. (1972) Some Properties of Special Functions Bivariate Distributions.* Sankhya-Series B. Vol. 34, p. 251-264.

4. *Saxena,R.K. and Sethi. P. L. (1973) Certain Properties of Bi-variate Distributions Associated with Generalized Hyper-geometric Functions.* Canadian Journal of Statistics Vol. 1, p. 171-180.

5. *Erdelyi, A. et al (1963) Higher Transendental Functions Vol. 1.* McGraw Hill Company, New York.

6. Erdelyi, A. et al (1954) *Tables of Integral Transforms* Vol. 1, McGraw Hill Company, New York.

7. ________________ (1954) *Tables of Integral Transforms* Vol. 2, McGraw Hill Company, New York.

8. Rogers, G. S. and Young. D. L. (1974) *Testing and Estimation When a Normal Covariance Matrix has Intra-class Structure of Arbitrary Order.* Communications in Statistics, Vol. 3, p. 343-359.

9. Mathai, A. M. and Saxena, R. K. (1971) *Extensions of An Euler Integral Through Statistical Techniques.* Mathematische Nachriten, Vol. 51, p. 1-9.

10. Mathai, A. M. (1970) *An Expansion of Meiger's G-Function in the Logarithmic Case with Applications.* Mathematische Nachriten Vol 48, p. 130-139.

11. Holla, M. S. (1968) *On a Non-Central Chi-square Distribution in the Analysis of Weapon Systems Effectiveness.* Metrika Vol. 14 p. 9-14.

12. de Waal, D. J. (1974) *Bayes Estimate of the Non-Centrality Parameters in Multivariated Analysis.* Communications in Statistics Vol. 3 p. 73-79.

Department of Mathematics
Sir George Williams University
Montreal, Canada

(Recibido en diciembre de 1974)