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# SOME TOPOLOGICAL EXTENSIONS OF PLANE GEOMETRY

*by*

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## ABSTRACT

Let $A$ be a closed subset of the plane. For a point $b$ in the plane whose distance to $A$ is $d(b,A)$ let $S(b,A)=\{z : |z-b| = d(b,A)\}$. Let $E(A)$ be the set of points $e$ for which $S(b,A)\cap A$ has at least two points. Let $\mathcal{I}'(A)$ be the set of open intervals $(a,b)$ for which there is an $e\in E(A)$ and a component of $S(e,A)-A$ with endpoints $a$ and $b$. The sets $E(A)$ and $\mathcal{I}'(A)$ are the central tools in this paper.

It is proven that the set of points equidistant between two mutually disjoint plane continua is always a connected one manifold. It is shown that the convex hull of a closed set $A$ can be written as the disjoint union of sets consisting of $A$, open intervals with endpoints in $A$ and open triangular regions with vertices in $A$, with the property that every map defined on $A$ extends to a map defined on the convex hull of $A$ that is linear on each of the open intervals and open triangular regions. It is shown that if $A$ is a continuum and $e, f \in E(A)$ are in the same bounded component of the complement of $A$ then there is a unique arc in $E(A)$ that joins $e$ to $f$. If $e \in E(A)$ is in the unbounded component of the complement of the continuum $A$ then there is a unique unbounded topological ray in $E(A)$ with endpoint $e$. It is shown that every plane continuum can be approximated from above by a plane continuum whose boundary consists of a finite number of simple closed curves that are contained in the union of $A$ and the $I$ in $\mathcal{I}'(A)$. Arcs in $E(A)$ and intervals in $\mathcal{I}'(A)$ intersect in a way that permits their use to develop a notion of tangent lines and normal arcs for arbitrary plane continua that complements the usual notions.

In this paper topological methods shall be used to extend some aspects of plane geometry that have their roots in analytic geometry. Since we shall restrict our interest to the plane, all sets will be assumed to be subsets of the plane unless otherwise indicated.

*Some notation and definitions :* If $A$ is a non-empty set and $p$ is a point then the distance from $p$ to $A$ $d(p,A)$ is defined to be the $\inf\{\,|p-a| : a \in A\}$. A point $p$ is equidistant from two non-empty sets $A$ and $B$ if $d(p,A) = d(p,B)$. The set of points equidistant from $A$ and $B$ shall be denoted by $E(A,B)$, $S(p,A)$ shall denote $\{x : |p-x| = d(p,A)\}$, and finally $E(A)$ shall denote $\{x : (S(x,A)) \cap A$ has at least two points$\}$. Central to all parts of the paper is the set $E(A)$, where $A$ is a closed set.

Aside from the usual elementary topological notions the paper depends on the topological characterization of the unit interval as a compact connected, metric space with exactly two-non-cutpoints and on the consequences of the fact that the plane is unicoherent. Recall that a topological space $X$ is unicoherent if whenever $X = A \cup B$ where $A$ and $B$ are closed connected sets then $A \cap B$ is connected.

In this section the main theorem states that the set of points equidistant from two closed mutually disjoint closed connected subsets of the plane is a one manifold.

In section 2 a family of mutually disjoint open intervals $\mathcal{I}(A)$ determined by a closed set $A$ and related to $E(A)$ is introduced. The main results assert that for each closed set $A$ there is a family $\mathcal{J}$ of mutually dispoint open intervals with endpoints in $A$ that contains $\mathcal{I}(A)$, and a family of mutually disjoint open triangular regions $\mathcal{T}$ with vertices in $A$ and edges in $\mathcal{J}$ such that : (i) The convex hull of $A$ is covered by $A$, the $J \in \mathcal{J}$, and the $T \in \mathcal{T}$, and (ii) If $f$ is any continuous complex valued function defined on $A$ then the extension of $f$ to the convex hull of $A$ that is linear on each $J \in \mathcal{J}$ and each $T \in \mathcal{T}$ is continuous. The topological part of the Shoenflies theorem follows automatically.

In section 3 the set $E(A)$ and the related family $\mathcal{J}(A)$ introduced in section 2 are studied for the purpose of developing a notion of tangent and normal lines to an arbitrary plane continuum $A$ that complements the usual notions developed for nice sets. Along the way it is shown that each point of $E(A)$ contained in the unbounded component of the complement of $A$ is the endpoint of a unique unbounded topological ray contained in $E(A)$, the intersection of a bounded component of the complement of $A$ and $E(A)$ is connected, any two points in the same component of $E(A)$ are the endpoints of a unique arc in $E(A)$, and if $J \in \mathcal{J}(A)$ separates two points with respect to the component of the complement of $A$ that contains $J$ then any arc in $E(A)$ that joins these points intersects $J$ at exactly one point.

In section 4 methods are developed to approximate a plane continuum $A$, without cutpoints by a continuum whose boundary is contained in $A \cup (\bigcup \{J: J \in \mathcal{J}(A)\})$ and is the union of a finite number of simple closed curves. It should be noted that the supporting lemmas will be used throughout the rest of the paper, particularly lemma 1.1 and its corollaries. Lemma 1.1 is well known, [4], however a proof is included here for completeness.

*THEOREM 1. Let A and B be non-empty mutually disjoint closed connected sets. Then E(A,B) is a one-manifold. That is, E(A,B) is homeomorphic to a simple closed curve or to the set of real numbers*

Theorem 1 shall be proven with aid of the following sequence of lemmas.

*LEMMA 1.1. Let* $C$ *be a closed set and let* $f: R^2 \to C$ *for which* $|p - f(p)| = d(p,C)$ *for* $p \in R^2$. *Then if* $p \in R^2$ *and* $q \in [p, f(p))$ *then* $f(q) = f(p)$.

*Proof:* $|p - f(p)| = |p - q| + |q - f(p)| \geq |p - q| + d(q,C) = |p - q| + |q - f(q)| \geq |p - f(q)| \geq d(p,C) = |p - f(p)|$. Therefore $|p - f(q)| = |p - q| + |q - f(q)|$. Therefore $q$ is on the interval between $p$ and $f(q)$ and on the interval between $p$ and $f(p)$. Therefore $f(p) \in [p, f(q)]$ or $f(q) \in [p, f(p)]$. Since $|p - f(p)| = d(p, C)$ and $f(q) \in C$ we conclude that $f(p) \in [p, f(q))]$. Since $q \in [p, f(p))$ we conclude that $f(p) \in [q, f(q)]$. Again since $f(p) \in C$ and $|q - f(q)| = d(q, C)$ it is clear that $f(q) = f(p)$.

*COROLLARY 1.1.1 : If* $C$ *is a non-empty closed set and* $|p - a| = d(p, C)$ *for some* $a \in C$ *then* $((p, a)) \cap E(C) = \phi$.

*COROLLARY 1.1.2 : Let* $C$ *be a closed set and let* $f: R^2 \to C$ *be such that* $|p - f(p)| = d(p,C)$ *for each* $p \in R^2$. *Then if* $x, y \in R^2 - C$ *we have*

$$([x, f(x))) \cap ([y, f(y))) = \phi$$

or

$$[x, f(x)) \subset [y, f(y)) \quad \text{or} \quad [y, f(y)) \subset [x, f(x)).$$

*Proof:* Suppose $p \in ([x, f(x)) \cap ([y, f(y)))$, then according to lemma (1) $f(x) = f(y)$. The conclusion follows.

*PROPOSITION 1.1.1. Let* $f$ *be as in corollary 1.1.2. Then* $E(C)$ *is the set of discontinuities of* $f$. *Furthermore if* $J = [a,b]$ *is an interval that does not intersect* $E(C)$ *then* $f$ *is monotone on* $J$, *consequently* $f(J)$ *is an arc with endpoints* $f(a)$ *and* $f(b)$ *or a single point.*

*DEFINITION :* For the remainder of this section fix $a: E(A,B) \to A$ and $b: E(A,B) \to B$ such that $a(x) \in A \cap S(x,A)$ and $b(x) \in B \cap S(x, B)$. Let

$L(x) = (a(x), x] \cup [x, b(x))$. Notice that $a$ and $b$ are not generally unique.

*LEMMA 1.2 : The set $E(A,B)$ is connected.*

*Proof :* Clearly $E(A,B)$ separates $A$ from $B$. Since the plane is unicoherent [10] some component $K$ of $E(A,B)$ separates $A$ from $B$.

To see that $K = E(A,B)$ let $x$ be an arbitrary point in $E(A,B)$ and notice that the connected set $\overline{L(x)}$ intersects both $A$ and $B$ and must accordingly intersect $K$. Letting $A \cup B$ replace $C$ in the corollary to 1,1.2 it follows that the only place that $\overline{L(x)}$ can intersect $E(A,B) \subset E(A \cup B)$ is at the point $x$. Since $K \subset E(A,B)$ we have $(\overline{L(x)}) \cap K \subset (\overline{L(x)}) \cap E(A,B) = \{x\}$. Therefore $x \in K$.

As an easy consequence of Theorem 28, p. 156 of [7] we have :

*LEMMA 1.3 : Let $C$ and $D$ be mutually disjoint plane continua that do not intersect the interior complementary domain of some simple closed curve $K$. Then there is a continuum in $K$ that contains $C \cap K$ and does not intersect $D$.*

*LEMMA 1.4 : $E(A,B)$ is a one-manifold.*

*Proof :* Let $e_0 \in E(A,B)$ and let $D$ be the open circular disk with center at $e_0$ and radius less than $(d(e_0, A))/2$. Let $J$ be the boundary of $D$. For each $e \in D \cap E(A,B)$ let $a'(e)$ be the point in $J \cap [e, a(e))$ and let $b'(e)$ be the point in $J$ $[e, b(e))$. If $e \in D \cap E(A,B)$ and we let $A \cup [a'(e), a(e)) \cup [a'(e_0), a(e_0))$, replace $C$ and $B \cup [b'(e), b(e)) \cup [b'(e_0), b(e_0))$, replace $D$ in lemma 1.3, then lemma 1.3 asserts that there are mutually disjoint arcs $M(e)$ and $N(e)$ in $J$ such that $a'(e), a'(e_0) \in M(e)$ and $b'(e), b'(e_0) \in N(e)$. Accordingly there exists mutually disjoint arcs $M$ and $N$ in $J$ such that

$$\{ a'(e) : \ e \in D \cap E(A,B) \} \subset M \quad \text{and}$$

$$\{ b'(e) : \ e \in D \cap E(A,B) \} \subset N .$$

Since $(E(A,B)) \cap D$ separates $(e_0 + a'(e_0))/2$ from $(e_0 + b'(e_0))/2$ in $D$ some component $K'$ of $(E(A,B)) \cap D$ does. Since the connected set $(L(e_0)) \cap D$ contains both $(e_0 + a'(e_0))/2$ and $(e_0 + b'(e_0))/2$, $(L(e_0)) \cap K' \neq \phi$. Therefore $e_0 \in K'$ and the situation depicted in figure 1 exists. In figure 1, $e_1$, $e_2 \in K'$, the circular arc $a'(e_2)a'(e_0)\,a'(e_1)$ is contained in $M$, the circular arc $b'(e_2)b'(e_0)\,b'(e_1)$ is contained in $N$, and $([a'(e_2)\,a'(e_0)\,a'(e_1))] \cup [b'(e_2)\,b'(e_0)\,b'(e_1)]) \cap E(A,B) = \phi$. Let $D_1$ be the open shadowed region

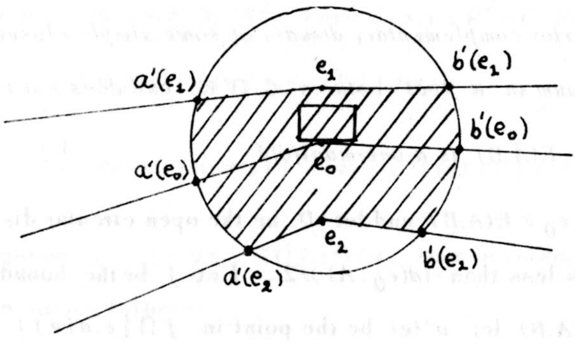


Figure 1

in figure 1. Let $K_1$ be the component of $K' \cap D_1$ that contains $e_0$. Clearly $K_1$ separates $(a(e_0) + e_0)/2$ from $(b'(e_0) + e_0)/2$ in $D_1$. Also if

$e \in (E(A,B)) \cap D_1$ then $(L(e)) \cap D_1$ separates $K_1$ and must therefore intersect $K_1$. Since $L(e)$ can intersect $K_1 \subset E(A,B)$ only at $e$ we conclude that $e \in K_1$. Therefore $D_1 \cap (E(A,B)) = K_1$. Since $L(e)$ separates $\overline{K_1} = K_1 \cup \{ e_1, e_2 \}$ and $(L(e)) \cap K_1 = \{ e \}$ for every $e \in K_1$ it follows that every $e \in K_1$ is a cut point of $K_1$. Therefore $\overline{D_1 \cap (E(A,B))}$ is a connected ,compact metric space with exactly two non cutpoints. $\overline{D_1 \cap (E(A,B))}$ is therefore a closed arc and $D_1 \cap E(A,B)$ is an open curve.

*COROLLARY 1.1. $E(A,B)$ is a simple closed curve or each point of $E(A,B)$ separates $E(A,B)$ into two components each of which is unbounded.*

*Proof:* The proof is straight forward and left to the reader.

*Remark.* If $0 < r < 1$ then $\{ x : d(x, A) = rd(x, B) \}$ is not in general a one manifold. For example let $A = \{ (x,y) : x^2 + y^2 = r^2 \}$ and $B = \{ (x,y) : x^2 + y^2 = 1 \}$ then $(0,0)$ is an isolated point of $\{ x : d(x, A) = rd(x, B) \}$.

*Question :* One might wonder what is true in higher dimensions.

If $A$ and $B$ are mutually disjoint closed connected subsets of $R^n$ then $E(A,B)$ is a closed connected set with an empty interior. In particular the dimension of $E(A,B) \leq n-1$. Since $E(A,B)$ separates every open set that it intersects it follows [5] that $E(A,B)$ has local dimension $\geq n-1$ at each point.

If $A = \{ (x,y,z) : x^2 + y^2 = 1$ and $z = 0 \}$ and $B = \{ (x,y,z) : x^2 + y^2 + z^2 = 9$, or $x = 0 = y$ and $1 \leq \|z\| \leq 3 \}$ then $E(A,B)$ fails to be a two manifold at the origin.

When is $E(A,B)$ and $n-1$ manifold ? Will "approximations" of $A$ and $B$, $A'$, $B'$ always be found so that $E(A', B')$ is an $(n-1)$ manifold ?

In [2] Morton Brown proves some interesting theorems about the closely related set $\partial_\varepsilon(A) = \{ x \in R^n : d(x, A) = \varepsilon \}$.

*Section 2 :* Let $A$ denote a closed set with convex hull $CH(A)$. Notice that if $\mathcal{J}$ is a continuous family of mutually disjoint open intervals each contained in the complement of $A$ and each having endpoints in $A$, then any map defined on $A$ can be extended linearly to each $J \in \mathcal{J}$ to obtain a map defined on $\cup\{J : J \in \mathcal{J}\} \cup A$. In this section such a family of open intervals is defined with the added property that every component of $CH(A) - (\cup\{ J : J \in \mathcal{J}(A) \} \cup A)$ is a "nice" open convex set. This allows for the extension of $\mathcal{J}(A)$ to a larger continuous family of disjoint family of disjoint open intervals having endpoints in $A$, $\mathcal{J}(A)$, such that each component of $CH(A) - (\cup\{ J : J \in \mathcal{J}(A) \} \cup A)$ is a triangular disk whose boundary consists of three elements of $\mathcal{J}(A)$ and three elements of $A$.

If a map $f$ defined on $A$ is extended linearly to each $J \in \mathcal{J}(A)$ and to each component of the complement of $CH(A) - (\cup\{ J : J \in \mathcal{J}(A) \} \cup A)$ then the extension is continuous. If $A$ happened to be a simple closed curve and $f$ happened to be a homeomorphism of $A$ onto the boundary of the circular unit disk then the extension automatically maps the bounded component of the complement of $A$ homeomorphically onto the interior of the unit disk.

*DEFINITION.* Let $A$ be a closed set. For each $e \in E(A)$ let $\mathcal{J}_A(e)$ be the set of open intervals $(a,b)$ for which there is a component of $S(e, A) - A$ with endpoints $a$ and $b$. Let $\mathcal{J}'(A) = \cup\{ \mathcal{J}_A(e) : e \in E(A) \}$. Let $\mathcal{J}(A)$ be the set of open intervals $(a,b)$, that do not intersect $A$, and for which there is a sequence $\{ (a_i, b_i) \}$ in $\mathcal{J}'(A)$ such that $a = \lim a_i$ and $b = \lim b_i$. We shall say that the sequence $\{ (a_i, b_i) \}$ converges to $(a,b)$ and if in addition

$(a,b) \not\in \mathcal{I}'(A)$, $(a,b)$ will be called a limiting interval. (Lemma 2.4 shall show that the condition that $(a,b)$ not intersect $A$ is redundant).

*LEMMA 2.1. Let $N$ and $M$ be distinct closed circular disks and let $K$ and $J$ be distinct open intervals contained in $N$ and $M$ respectively. If $K \cap J \neq \phi$, then the interior of $N$ contains and endpoint of $J$ or the interior of $M$ contains an endpoint of $K$.*

*Proof.* : The cases $M \subset N$, $N \subset M$, $N \cap M = \{x\}$, and $N \cap M = \phi$ are clear. In all other cases the boundary of $M$ intersects the boundary of $N$ at exactly two points say $p$ and $q$. Let $L = [p,q]$. Write $M \cup N$ as the union of two topological two-cells $M'$ and $N'$ where $M' \subset M$, $N' \subset N$ and $M' \cap N' = L$. Consider and dismiss the cases $J = L$ and $K = L$. The cases $J \cap L \neq \phi$, $K \cap L \neq \phi$, $J \subset M' - L$, and $K \subset N' - L$ once considered are clear. The remaining case $J \subset N' - L$ and $K \subset M' - L$ is not possible since by hypothesis $J \cap K \neq \phi$.

*LEMMA 2.2 : If $J$, $K$ are distinct intervals in $\mathcal{I}(A)$ then $J \cap K = \phi$.*

*Proof* : If $J, K \in \mathcal{I}_A(e)$ for some $e \in E(A)$ then clearly $J \cap K = \phi$. If $J \in \mathcal{I}_A(e_0)$ and $K \in \mathcal{I}_A(e_1)$ where $e_0 \neq e_1$ lemma 2.1 asserts that either $J \cap K = \phi$ or some endpoint of $J$ is in the interior of the disk bounded by $S(e_1,A)$ or some endpoint of $K$ is in the interior of the disk bounded by $S(e_0,A)$. Since the endpoints of $J$ and $K$ are in $A$ the last two cases are impossible, it follows that $J \cap K = \phi$.

If either $K$ or $J$ is a limiting interval then clearly $K \cap J \neq \phi$ implies $K' \cap J' \neq \phi$ for a pair of intervals $K'$ and $J'$ of $\mathcal{I}'(A)$.

*LEMMA 2.3 : Let $A$ be a closed set, let $B$ be a compact set, and let $M = (\bigcup \{J : J \in \mathcal{I}_A(e)$* for some *$e \in B \cap (E(A))\} \cup A)$. Then $M$ is closed.*

*Proof:* Let $x_o$ be some limit point of $M$. If there are sequences $\{x_i\}, \{a_i\}, \{b_i\}, \{J_i\}$ and $\{e_i\}$ where each $x_i \in (a_i, b_i) = J_i \in \mathcal{J}_A(e_i)$ the $x_i$'s converge to $x_o$, the $a_i$'s converge to a point $a_o$, the $b_i$'s converge to a point $b_o$, each $e_i$ is in $B$, and the $e_i$'s converge to some point $e_o$. Since the distance function $d$ is continuous $d(e_o, A) = \lim d(e_i, A) = \lim |e_i - a_i| = \lim |e_i - b_i| = |e_o - a_o| = |e_o - b_o|$. Therefore $e_o \in (E(A)) \cap B$. If $e_i = e_o$ for all but a finite number of $i$ then clearly $x_o \in (a_o, b_o) = (a_i, b_i) \subset M$ for large $i$. We may therefore assume that the $e_i$'s are distinct. Let $C_1$ and $C_2$ be the components of $S(e_o, A) - \{a_o, b_o\}$. If $C_1$ or $C_2$ does not intersect $A$ then $x_o \in (a_o, b_o) \subset M$ and our proof is finished. To see that this is indeed the case let $x_1 \in C_1$, let $x_2 \in C_2$, and let $L = [x_1, e_o] \cup [e_o, x_2]$. Now for sufficiently large $i$ either $[e_i, a_i]$ intersects $L$ or $[e_i, b_i]$ intersects $L$, thus by Corollary 1.1.2 either $x_1 \notin A$ or $x_2 \notin A$. Since $x_1$ and $x_2$ were chosen to be arbitrary points in $C_1$ and $C_2$ it follows that either $C_1$ or $C_2$ does not intersect $A$.

LEMMA 2.4 : *Let $A$ be a closed set and let $\{a_i\}, \{b_i\}, \{J_i\}$, and $\{c_i\}$ be sequences such that $(a_i, b_i) = J_i \in \mathcal{J}'(A)$, $c_i \in J_i$, $\lim a_i = p_1$, $\lim b_i = p_2$, and $\lim c_i = p_3 \in A$. Then $p_3 = p_2$ or $p_3 = p_1$.*

*Proof:* If $p_1 = p_2$ the lemma is trivial. Assume $p_1 \neq p_2$ and let $L$ be the line that contains the points $p_1, p_2$ and $p_3$. Let $e_i \in E(A)$ be such that $J_i \in \mathcal{J}_A(e_i)$. Notice that $L$ cannot contain $e_i$ for more than a finite number of $i$.

Write the complement of $L$ as the union of two open half planes $U$ and $V$ where $e_i \in U$ for infinitely many $i$. If the sequence $e_i$ has a cluster point $e$

then by lemma 2.3 $(p_1, p_2) \in \mathcal{J}'(A)$ . Therefore $p_3 \in A \cap [p_1, p_2] = \{p_1, p_2\}$ .

If $\{e_i\}$ has no cluster point then every point in $U$ is contained in the interior of some disk $D_i$ of radius $|e_i - a_i|$ and center at $e_i$, so that $A \cap U = \phi$. For $n = 1, 2$, and $3$ let $P_n$ be the ray in $\overline{U}$, perpendicular to the line $L$, that contains the point $p_n$. Since every point on $P_n$ is closer to $p_n$ than to any other point of $A$ it follows from Corollary 1.1.2 that $[e_i, a_i] \cup [e_i, b_i]$ cannot intersect $P_1, P_2$, or $P_3$. Since $p_1 = \lim a_i$ and $p_2 = \lim b_i$ this is impossible unless $P_3$ coincides with $P_1$ or with $P_2$.

*COROLLARY 2.4.1 : If $(a_i, b_i) \in \mathcal{J}'(A)$, $\lim a_i = a$ , $\lim b_i = b$, and $a \neq b$ then $(a,b) \in \mathcal{J}(A)$ .*

*COROLLARY 2.4.2 : Lemma 2.4 holds even if $J_i \in \mathcal{J}(A)$ for each $i$ .*

*COROLLARY 2.4.3 : Let $A$ be a compact set and let $M = \cup\{J : J \in \mathcal{J}(A)\} \cup A$. Then $M$ is closed.*

*LEMMA 2.5 : Let $A$ be a compact set and let $B$ be the boundary of the convex hull of $A$, $CH(A)$ . Then each component of $B - A$ is some $J \in \mathcal{J}(A)$ . Furthermore if $J \in \mathcal{J}(A)$ is a limiting interval then $J$ is a component of $B - A$.*

*Proof :* Let $K = (a,b)$ be a component of $B - A$ and let $L$ be the line that contains $K$ . Write the complement of $L$ as the union of two open half planes $U$ and $V$ where $A \cap U = \phi$. Let $P_a$ and $P_b$ be the rays in $\overline{U}$, that are perpendicular to $L$ , such that $P_a$ has endpoint $a$ and $P_b$ has endpoint $b$ . Let $C = \{x \in A : |x - a| \leq |x - b|\}$ and let $D = \{x \in A : |x - b| \leq |x - a|\}$ . Since $E(C,D)$ separates $P_a$ from $P_b$ there exists a sequence $\{e_i\}$ in the convex region bounded by $P_a$, $P_b$, and $K$ such that each $e_i \in E(C,D)$ and $\lim |e_i| = \infty$ . Let $a_i \in S(e_i, C) \cap C$ and let $b_i \in S(e_i, D) \cap D$ . Clearly

$a = \lim a_i$ and $b = \lim b_i$, which shows that $e_i \in E(A)$ for large $i$. We may therefore, for large $i$, find $J_i = (c_i, d_i) \in \mathcal{J}_A(e_i)$ such that $c_i \in C$ and $d_i \in D$. Again we have $\lim c_i = a$ and $\lim d_i = b$. Thus by corollary 2.4.1 $K = (a,b) \in \mathcal{J}(A)$.

Let $J$ be a limiting interval and let $L$ be the line that contains $J$. Suppose $J = \lim J_i$ and each $J_i \in \mathcal{J}_A(e_i)$. According to lemma 2.3 $\lim |e_i| = \infty$. Therefore every point in one of the components of the complement of $L$ must be contained in some $S(e_i, A)$. This clearly shows that $J$ is contained in $B$.

*THEOREM 2.1 : Let $A$ be a closed set and let $M = \bigcup\{J : J \in \mathcal{J}(A)\} \cup A$. If $f: A \to R^2$ is continuous and $g$ is the extension of $f$ to $M$ that is linear on each $J \in \mathcal{J}(A)$ then $g$ is continuous.*

*Proof :* Let $m \in M$ and let $\{m_i\}$ be a sequence in $M$ that converges to $m$. We may assume without loss of generality that either each $m_i \in A$ or each $m_i \in (a_i, b_i) = J_i \in \mathcal{J}(A)$, $a = \lim a_i$ exists, and $b = \lim b_i$ exists. If each $m_i \in A$ then clearly $m \in A$ and $\lim g(m_i) = \lim f(m_i) = f(\lim m_i) = f(m) = g(m)$. We assume then that $m_i = a_i t_i + b_i(1 + t_i)$ for some $t_i$ with $0 \leq t_i \leq 1$.

If $m \in A$ then by lemma 2.4 $m = a$ or $m = b$. Assume $m = a$. Then $a = b$ or $\lim t_i = 1$. In either event

$$g(m_i) = g((a_i t_i + b_i(1 - t_i)) = f(a_i)t_i + f(b_i)(1 - t_i)$$

$$\lim [\, f(a_i)t_i + f(b_i)(1 - t_i)\,]$$

If $a \neq b$ and $m \notin (a,b)$ which by corollary 2.4.1 is in $\mathcal{J}(A)$. Write $m = (a - b)t + b$ where $t = \lim t_i$. Accordingly $g(m) = f(a) - f(b))t + f(b) = \lim((f(a_i) -$

$f(b_i)) t_i + f(b_i)) = \lim g(m_i)$ .

*DEFINITION :* For $A$ a closed set let $E'(A)$ be the set of points in $E(A)$ for which $S(e, A) \cap A$ has at least three points. Let $C(e)$ be the interior of the convex hull of $S(e, A) \cap A$ . Notice that the boundary of $C(e)$ is the simple closed curve $\cup \{ J : J \in \mathcal{J}_A(e) \} \cup (S(e, A) \cap A)$ .

*LEMMA 2.6. Let $A$ be a compact set, let $p$ be a point in the convex hull of $A$, and let $q = (a, b) \in J_o \in \mathcal{J}(A)$. If the half open interval from $p$ to $q$, $[p, q)$ intersets no $J \in \mathcal{J}(A)$ then there is an $e \in E(A)$ such that $J_o \in \mathcal{J}_A(e)$ .*

*Proof :* Let $L$ be the line that contains $J_o$ and write the complement of $L$ as the union of two open half planes $U$ and $V$ where $p \in U$ . Since $p$ is in the convex hull of $A$ there is a $z' \in A \cap U$ . Let $\{ J_i \}$ be a sequence in $\mathcal{J}'(A)$ that converges to $J_o$ . Notice that $J_i$ cannot be contained in $U$ for more than a finite number of $i$ since otherwise some $J_i$ would intersect $[p, q]$ . After barring the possibility that $J_i = J_o$ for some positive integer $i$ it follows that $V$ must contain an endpoint of some $J_k$ say $z$ , since $J_o$ is contained in the interior of the convex hull of $\{ z', z, a, b \}$ the result follows from lemma 2.5.

*THEOREM 2.2 : Let $A$ be a compact set with convex hull $CH(A)$ and let $M = \cup \{ J : J \in \mathcal{J}(A) \} \cup A$ . Then each component of $CH(A) - M$ is a $C(e)$ for some $e \in E'(A)$ .*

*Proof :* Let $K$ be a component of $CH(A) - M$. In order to prove that $K$ is some $C(e)$ we consider several cases.

*Case 1 :* $A$ is contained in an interval. In this case lemma 2.5 assures us

$CH(A) = M$ and there is nothing to prove.

*Case 2 :* The boundary of $K$ is entirely contained in $A$. In this case let $e$ be a point in $K$ whose distance to $A$ is maximal. Clearly $e \in E(A)$ and the boundary of $K$ is in fact $S(e,A)$. It follows that $K = C(e)$.

*Case 3 :* The boundary of $K$ is not entirely contained in $A$. In this case there is a polygonal arc $P$ that intersects both $K$ and the complement of $K$ and does not intersect $A$. Lemma 2.5 and corollary 2.4.3 guarantee that $K$ is open so that $K \cap P$ contains a half open interval $[p,q)$ with $q$ not in $K$. It follows from corollary 2.4.3 that $q$ is on some $J = (a,b) \in \mathcal{J}(A)$. Lemma 2.6 yields that $J$ is in $\mathcal{J}_A(e)$ for some $e \in E(A)$.

Let $L$ be the line that contains $J$ and write the complement of $L$ as the union of two half planes $U$ and $V$ where $p \in U$.

*Case 3.1 :* $\bar{U}$ contains an $e_o \in E(A)$ for which $J \in \mathcal{J}_A(e_o)$. Since $CH(A)$ and therefore $A$ intersects $U$ there is an $e' \in E(A)$, such that $J \in \mathcal{J}_A(e')$, whose distance to $J$ is maximal. If $e' \in E'(A)$ then $p \in C(e')$ and it follows that $K = C(e')$. To see that this is indeed the case suppose $S(e', A) \cap A = \{a, b\}$ and let $W_a$ and $W_b$ be mutually disjoint compact neighborhoods of $a$ and $b$. Let $\{e_i\}$ be a sequence in $E(W_a \cap A, W_b \cap A)$ that converges to $e'$ and contains no points in the perhaps degenerate triangular region bounded by $[a,e']$, $[e', b]$, and $[a,b]$. Now for large $i$, $d(e_i, W_a \cap A) = d(e_i, A) = d(e_i, W_b \cap A)$ so that there is a sequence $\{(a_i, b_i)\} = \{J_i\}$ such that $J_i \in \mathcal{J}_A(e_i)$ for large $i$, $a_i \in W_a \cap A$ and $b_i \in W_b \cap A$. Now since $(a,e') \cup [e',b)$ cannot intersect $(a_i, e_i] \cup [e_i, b_i)$ for large $i$, it follows that for large $i$ either $a_i \in U$ or $b_i \in U$. Since the $J_i$ converge to $J$ this would force the $J_i$ to intersect $[p,q)$

for large $i$, contradicting the fact that $[p, q) \subset K$.

*Case 3.2:* There is no $e \in E(A) \cap \overline{U}$ such that $J \in \mathcal{J}_A(e)$. In this case let $e' \in E(A) \cap V$ such that $J \in \mathcal{J}_A(e')$ and the distance from $e'$ to $J$ is minimal. The proof now proceeds similarly to the proof in case 3.1 with the noted exception that the sequence $\{e_i\}$ is chosen inside the interior of the triangle bounded by $[a,e']$, $[e',b]$, and $[a,b]$.

***DEFINITION:*** **For $A$ a closed set let $\mathcal{J}(A) = \bigcup_{n=1}^{\infty} \mathcal{J}_n$ be a family of mutually disjoint open intervals defined using finite induction as follows: Let $\mathcal{J}_1(A) = \mathcal{J}(A)$. Suppose that $\mathcal{J}_n(A)$ has been defined. For each component $K$ of the complement of $CH(A) - \bigcup\{J : J \in \mathcal{J}_n(A)\} \cup A$, that is not a circular disk or an open triangular region, add two intervals to the family $\mathcal{J}_n(A)$, $(a,p)$ and $(b,p)$, where $(a,b) \in \mathcal{J}_n(A)$ is contained in the boundary of $K$ and has maximal diameter, and $p$ is a point of $A$ on the boundary of $K$ whose distance to the perpendicular bisector of $a$ and $b$ is minimal; if $K$ is a circular disk add one interval $J = (a,b) \subset K$ where $|a-b|$ is maximal.**

LEMMA 2.7. *Each component of $CH(A) - \bigcup\{J : J \in \mathcal{J}(A)\} \cup A$ is a triangular open disk whose boundary is the union of three $J$ in $\mathcal{J}(A)$.*

*Proof:* Let $K$ be a component of $CH(A) - \bigcup\{J : J \in \mathcal{J}(A)\} \cup A$. For each positive integer $n$ let $K_n$ be the component of $CH(A) - \bigcup\{J : J \in \mathcal{J}_n(A)\} \cup A$ that contains $K$. Notice that if $K_n = K_{n+1}$ for some positive integer $n$ then $K = K_n$ and the lemma follows. To see that this is indeed the case suppose not. Then for each positive integer $n$, $K_n$ contains two open intervals $(a_n, p_n)$ and $(b_n, p_n)$ in $\mathcal{J}_{n+1}(A)$ where $J_n = (a_n, b_n) \in \mathcal{J}_n(A)$ and $||b_n - a_n||$ is maximal. Now according to theorem 2.2 $K_1$ is contained in $C(e)$ for some $e \in E'(A)$. Sininterval $[p, q) \subset D - g(B)$ where $q \in (g(B)) \cap D$. Clearly $q = g(z)$ for some $z \in J_z = (a,b) \in \mathcal{J}(A)$. Let $L$ be the arc in $S^1$ that joins $g(a)$ to $g(b)$ and is such that the bounded complementary domain of $(g(a), g(b)) \cup L$, $D_1$ contains $[p,q)$. Let $B_1$ be the bounded complementary domain of $J_z \cup g^{-1}(L)$. If $B_1$ contains a sequences of points $\{z_i : i = 1, 2, \ldots\}$ where each $z_i \in J_i \in \mathcal{J}(A)$ and $\lim z_i = z$ then the $J_i$ converge to $J_z$ and the $g(J_i)$ converge to $g(J_z)$. Since $g(J_i) \subset D_1$ for each $i$ it follows that $(g(J_i)) \cap [p,q) \neq \phi$ for large $i$, contradicting the assumption that $(g(D)) \cap [p,q) = \phi$. If no such sequence exists then there is a point $z'$ in $g^{-1}(L)$ such that $(a, z')$ and $(b, z') \in \mathcal{J}(A)$ and the open triangular region bounded by $J_z$, $[a, z']$, and $[b, z']$, $Q$ is a component of $CH(A) - \cup\{J : J \in \mathcal{J}(A)\} \cup A$. Clearly $(g(Q)) \cap [p,q) \neq \phi$.

To see that $g$ is *1-1* let $u,v$ be distinct elements of $B$. If $u$ and $v$ are contained in the same $J \in \mathcal{J}(A)$ or in the closure of the same component of $CH(A) - \cup\{J : J \in \mathcal{J}(A)\} \cup A$ then clearly $g(u) \neq g(v)$. If $u \in J \in \mathcal{J}(A)$ and $v \notin \mathcal{J}(A)$ and $v \notin J$ then $A \cup J$ contains a simple closed curve $B_2$ whose bounded complementary domain contains $v$. Clearly, $g(v)$ is contained in the bounded complementary domain of $g(B_2)$. If $u$ is in some component of $CH(A) - \cup\{J : J \in \mathcal{J}(A)\} \cup A$, $K$, and $v \notin K$ then as a consequence of the Jordan curve theorem, some $J \in \mathcal{J}(A)$ contained in $K$ separates $u$ from $v$ in $B$. Clearly $g(J)$ separates $g(u)$ from $g(v)$ in $D$.

*Section 3 :* A theorem of Motzkin ([8] and [9]) states that a closed set $A$ in convex if and only if $E(A) = \phi$. That is, $A$ is convex if and only if there is a continuous function $M : R^2 \to A$ such that $|m(x) - x| = d(x, A)$ for every $x$ in the plane. If $m$ is such a function and $x \notin A$ then the ray that passes through $x$ and has $m(x)$ as an endpoint is said to be normal to the convex set $A$ at the point $m(x)$. If $L$ is a ray that is normal to $A$ at the point $b$ then the line perpendicular to $L$ that passes through the point $b$ is said to be a tangent line to $A$ at the point $b$.

It is not the purpose of this paper to survey or develop the standard notions of normal or tangent lines for "nice" closed sets but rather to develop similar notions that will enable us to deal with closed sets that are not nice. The development is not meant to replace the standard treatment but to work in conjunction with the usual concepts. In fact the substitutes to be developed for tangent and normal lines are most meaningful only in the cases where the usual notions are least meaningful.

It should be noted that some of the theorems in the section were stated without proof in [1] for the purpose of adding geometric intuition to an apparently non-geometric development. For completeness they will be restated and proven here.

*DEFINITION :* For $e \in E(A)$ let $\mathcal{L}_A(e)$ be the set of $(a,e] \cup [e,b)$ such that $(a,b) \in \mathcal{I}_A(e)$.

*DEFINITION :* If $e$ is a point and $B$ is a set whose closure does not contain $e$ let $T_e(B)$ be the smallest closed set that contains $B$, does not contain $e$, and has a connected complement. If $B$ is a bounded set let $T(B)$ be the smallest compact set that contains $B$ and has a connected complement.

*Notation :* For the remainder of section 3 $A$ will denote a fixed continuum. For each bounded component of the complement of $A$, $K$, let $e(K)$ be a point in $K$ whose distance to $A$ is maximal and let $T_K = T_{e(K)}$. If $K$ is the unbounded component of the complement of $A$ let $T_K = T$ .

*DEFINITION :* Let $\leq$ be a relation defined on $E(A)$ by $e_1 \leq e_2$ if $e_1$ and $e_2$ are both on the same component $K$ of the complement of $A$ and either $e_1 = e(K)$ or there is an $L \in \mathfrak{L}_A(e_1)$ such that $T_K(L \cup A)$ contains the point $e_2$. For $e \in E(A)$ let $L_e = \{x : x \leq e\}$.

*LEMMA 3.1 : The relation $\leq$ is a partial order, furthermore if $e \in E(A)$ then the set $L_e$ is totally ordered by the relation $\leq$ .*

*Proof :* The proof es straight forward and is left to the reader.

*LEMMA 3.2 : If $e \in E(A)$ is in the unbounded complementary component of $A$ and $A$ is a simple closed curve then $L_e$ is an unbounded topological ray with endpoint $e$ .*

*Proof :* Let $V_e \in \mathfrak{L}_A(e)$ with endpoints $a$ and $b$ . Let $w$ and $z$ be distinct points common to the boundary of the convex hull of $A$ and to $A$ . According to [5] there exists rays $R_w$ and $R_z$ with endpoints $w$ and $z$ such that every point in $R_w$ is closer to $w$ than to any other point of $A$ and every point in $R_z$ is closer to $z$ than to any other point in $A$ . Let the closed arc that is the intersection of the boundary of $T(V_e \cup A)$ and $A$ be denoted by $A_1$. Clearly $w$ and $z$ are on $A_1$. Let $A_2$ be the open arc contained in $A_1$ with endpoints $w$ and $z$ . Then $A_1 - A_2$ is the union of two (possibly degenerate) closed arcs $A_3$ and $A_4$ . According to Theorem 1 $E(A_3, A_4)$ is a one manifold. Since $E(A_3, A_4)$ is unbounded and therefore not a simple closed curve it is, by the corollary to theorem 1, the unbounded homeomorphic image of the set of real numbers. Clearly $L_e$ is a topological ray contained in $E(A_3, A_4)$ with endpoint $e$ .

*LEMMA 3.3 : If $e \in E(A)$ is in a bounded component of the complement of $A$, $K$, and $A$ is a simple closed curve then either $e = e(K)$ or $L_e$ is a closed arc from $e$ to $e(K)$.*

*Proof :* Suppose $e \neq e(K)$. Let $V_e \in \mathcal{L}_A(e)$ with endpoints $a$ and $b$, and let $V_K \in \mathcal{L}_A(e(K))$ with endpoints $c$ and $d$. Let $D$ be the simple closed curve in $A \cup V_e \cup V_K$ that contains both $e$ and $e(K)$. Then $D-(V_e \cup V_K)$ is the disjoint union of $A_1$ and $A_2$ where each $A_i$ is a closed arc or a single point. Theorem 1 assures us that $E(A_1, A_2)$ is a one manifold. Clearly $L_e$ is the intersection of $E(A_1, A_2)$ and $T(D)$, which is a closed arc.

*THEOREM 3.1 : Let $e \in E(A)$. Then $L_e$ is the unbounded homeomorphic image of the non-negative real numbers if $e$ is in the unbounded component of the complement of $A$. If $e$ is in a bounded component of the complement of $A$ then $L_e$ is an arc from $e$ to $e(K)$.*

*Proof :* The theorem shall be proven in the case $e$ is in the unbounded component of the complement of $A$. The other case is similar.

As is well known there is a sequence of simple closed curves $\{D_n\}$ such that $D_{n+1} \subset T(D_n)$ for each positive integer $n$ and $T(A) = \bigcap_{n=1}^{\infty} T(D_n)$. We also make the assumption that no $T(D_n)$ intersects the interior of $S(e, A)$.

For each positive integer $n$ let $D_n$ replace $A$ in lemma 3.2 to obtain a topological ray $L_e(n)$. It is straight forward to show thay $L_e$ is the topological limit superior of the $L_e(n)$. (The set of points such that every neighborhood intersects $L_e(n)$ for infinitely many $n$). Therefore $L_e$ is closed connected , unbounded and, by lemma 3.1, totally ordered by the relation $x \leq y$ if $x = e$ or $x$ separates $e$ from $y$.

$L = (c,d)$ then the result follows from corollary 1.1.1. Suppose then also that $L \cup (c,d)$ is the boundary of a triangular disk $T$. Since $L \cup A$ separates $a$ from $b$ it seems clear that the interior of $T$ must contain some point of $L(a,b)$, say $p$. Write $L(a,b) = L(a,p) \cup L(p,b)$. Since $L(a,p) \cap (L \cup (c,d)) \neq \phi$ and $L(p,b) \cap (L \cup (c,d)) \neq \phi$ and $L$ contains only one point of $E(A) \supset L(a,b)$, it follows that $L(a,b)$ must intersect $(c,d)$. To see that $L(a,b) \cap (c,d)$ has exactly one point let $e,f \in L(a,b) \cap (c,d)$, $V \in \mathcal{L}_A(e)$ and $V' \in \mathcal{L}_A(f)$ such that $V \cup A$ and $V' \cup A$ each separate $a$ from $b$. Since neither $V$ or $V'$ can intersect the interior of $T$ it follows that if $V \neq V'$ there is an arc $Q$ from $a$ to $b$ in the complement of $A$ such that $V \cap Q = \phi$ or $V' \cap Q = \phi$. Accordingly $V = V'$ and $e = f$.

*DEFINITION* : If $J$, $J_1$, and $J_2 \in \mathcal{J}(A)$ are all on the same component of the complement $K$ of $A$ then $J$ will be said to be between $J_1$ and $J_2$ if $J \cup A$ separates $J_1$ from $J_2$ and either $K$ is bounded or $K$ is unbounded and $J \subset T(J_1 \cup A) \cup T(J_2 \cup A)$.

*LEMMA 3.4 : If $J$ is between $J_1$ and $J_2$, $J \in \mathcal{J}_A(e)$, $J_1 \in \mathcal{J}_A(e_1)$, and $J_2 \in \mathcal{J}_A(e_2)$ then $e \in L(e_1, e_2)$.*

*Proof:* The proof is straight forward and is left to the reader.

*DEFINITION* : Let $W$ be the set of $(c,d)$ such that there is an $e \in E(A)$ and distinct $J, K \in \mathcal{J}_A(e)$ such that $c \in E(A) \cap J$ and $d \in E(A) \cap K$. For $a,b$ on the same component of $E(A)$ let $L'(a,b)$ be the arc $(L(a,b) - \bigcup \{L(c,d): c,d \in L(a,b) \text{ and } (c,d) \in W\}) \cup (\bigcup \{[c,d] : c,d \in L(a,b) \text{ and } (c,d) \in W\})$.

*LEMMA 3. 5 : Let $L(a,b)$ be an arc in $E(A)$ with $a \in J_1 \in \mathcal{J}(A)$ and $b \in J_2 \in \mathcal{J}(A)$. Then $L(a,b)$ and $L'(a,b)$ each intersect each $J \in \mathcal{J}(A)$ between*

*$J_1$ and $J_2$ at exactly one point. Furthermore $L'(a,b)$ intersects only those $J \in \mathcal{J}(A)$ that are between $J_1$ and $J_2$.*

*Proof:* The proof is straightforward and is left to the reader.

*Section 4: Approximating plane continua.* As in section 3 $A$ will denote a fixed plane continuum. In order to simplify the statements and proofs of some of the theorems it will be assumed that $A$ has no cutpoints.

The next lemma is an extension of lemma 3.5.

LEMMA 4.1: *If $J, K \in \mathcal{J}(A)$ and there is an $L$ between $J$ and $K$ then there is an arc $S$, from a point $a$ on $J$ to a point $b$ on $K$, that intersects every $L$ between $J$ and $K$ at exactly one point and intersects no other $L$ in $\mathcal{J}(A)$.*

*Proof:* Let $U$ be the component of the complement of $A$ that contains $J$ and $K$. Write $U - (J \cup K) = U_1 \cup U_2 \cup U_3$ where $J \subset \bar{U}_1$, $J \cup K \subset \bar{U}_2$, $K \subset \bar{U}_3$, and each $U_i$ is a component of $U - (J \cup K)$. If both $\bar{U}_1$ and $\bar{U}_3$ intersect $E(A)$ then there is an arc in $E(A)$ that joins a point in $\bar{U}_1$ to a point in $\bar{U}_3$. Such an arc must intersect both $J$ and $K$, in which case the result follows from lemma 3.5. Assume then that $\bar{U}_1$ does not intersect $E(A)$. According to theorem 3.1 $U_1$ is bounded. Since $U_2$ is also bounded, $J$ is not contained in the boundary of the convex hull of $A$, and is, according to lemma 2.5, not a limit interval. If $J \in \mathcal{J}_A(e)$ for some $e$ in $U_3$ then let $S$ be the closed subinterval of the closed interval joining $e$ to the midpoint of $J$, that joins the midpoint of $J$ to a point of $K$. Again the result follows.

The only remaining possibility is that $J \in \mathcal{J}_A(e_1)$ for some $e_1$ in $U_2$. If $K$ contains a point of $E(A)$, $e_2$, let $S$ be the union of the closed interval from the midpoint of $J$ to $e_1$ and $L'(e_1, e_2)$. Otherwise $K \varepsilon \mathcal{I}_A(e_1)$ for some $e_2 \varepsilon U_2$. In this case let $S$ be the union of the closed interval from $e_2$ to the midpoint of $K$, the arc $L'(e_1, e_2)$, and the closed interval from $e_1$ to the midpoint of $J$.

*DEFINITION :* For $J$, $K$, $\varepsilon \mathcal{I}(A)$ let $D(J, K)$ be the union of those $L$ in $\mathcal{I}(A)$ between $J$ and $K$ and those $C(e)$ (as in theorem 2.2) whose boundary contains an $L$ between $J$ and $K$.

*THEOREM 4.1 : If $J$ and $K$ are in $\mathcal{I}(A)$ and there is an $L$ between $J$ and $K$ then $D(J,K)$ is the interior of a closed topological two cell.*

*Proof :* The complement of $A$ contains a closed arc $S$ from a point on $J$ to a point on $K$ that intersects every $L$ between $J$ and $K$ at exactly one point and intersects no other $L$ in $\mathcal{I}(A)$. As a consequence of corollary 2.4.1 two monotone maps $a$ and $b$, defined on $S$, may be found such that if $s \varepsilon S \cap L$ where $L \varepsilon \mathcal{I}(A)$ then $L = (a(s), b(s))$ and if $S'$ is a component of some $S \cap C(e)$ then $a$ and $b$ map $S'$ into the boundary of $C(e)$.

Clearly the boundary of $D(J, K)$ is the union of the arcs $J$, $a(s)$, $K$ and $b(S)$.

*COROLLARY 1 : Let $J_o = (a,b) \varepsilon \mathcal{I}(A)$ be contained in the boundary of the convex hull of $A$ and let $\mathcal{I}' \subset \mathcal{I}(A)$ be finite. If each $J \varepsilon \mathcal{I}'$ is contained in $T(J_o \cup A)$ and there is a $K$ between $J$ and $J_o$ then $M_o = \cup \{D(J_o, J) : J \varepsilon \mathcal{I}'\}$ is the interior of a closed topological two-cell. Furthermore, $CH(A) = \overline{M}_o \cup V$ where $V$ is a closed topological two-cell for which $\overline{M}_o \cap V$ is a closed arc from $a$ to $b$.*

*COROLLARY 2 : Let $J_o \varepsilon \mathcal{I}(A)$ and let $\mathcal{I}' \subset \mathcal{I}(A)$ be finite. If for each*

*$J \in \mathcal{J}'$ there is a $J' \in \mathcal{J}(A)$ between $J$ and $J_o$ then $M_o = \bigcup \{D(J_o, J) : J \in \mathcal{J}'\}$ is the interior of a closed topological two-cell.*

*LEMMA 4.2 : Let $\mathcal{J}$ be an infinite subset of $\mathcal{J}(A)$ each of whose members has diameter greater than or equal to some fixed positive number $\delta$. Then there are $J$ and $L$ in $\mathcal{J}$, both in the same component $K$ of the complement of $A$, such that $T_K(J \cup A) \supset L$.*

*Proof:* **Since $\mathcal{J}$ is infinite there is an infinite sequence $\{J_i\}$ in $\mathcal{J}$ that converges to a $J_o$ in $\mathcal{J}(A)$. Clearly there is a $J'$ in $\mathcal{J}(A)$ for which $D(J_o, J')$ contains $J_i$ for infinitely many $i$. Given any three such $J_i$, none of which contains $e(K)$, some pair out of the three will satisfy the conclusion of the theorem.**

*LEMMA 4.3 : If $J = (a,b) \in \mathcal{J}(A)$ is contained in a component $K$ of the complement of $A$, $e(K) \not\in J$, and $T_K(J \cup A)$ contains no $L \in \mathcal{J}(A)$ other than $J$, then $\overline{T_K(J \cup A) - T_K(A)} \cap A$ is a circular arc from $a$ to $b$.*

*Proof:* **Let $U$ and $V$ be the components of $K - J$ where $e(K) \in U$ if $K$ is bounded and $U$ is unbounded if $K$ is unbounded. Notice that $J$ is not a limit interval since if $J = \lim J_i$ then for large $i$ either $J_i \subset T_K(J \cup A)$ or $J \subset T_K(J_i \cup A)$. The hypothesis disallows the first possibility and the second possibility implies that $J$ is not contained in boundary of the convex hull of $A$, a must for limit intervals.**

**Now consider the following two cases :**

*Case I:* **There is an $e \in (E(A)) \cap \overline{V}$ such that $J \in \mathcal{J}_A(e)$. In this case let $e_o$ be such an $e$ whose distance to $J$ is maximal. Write $S(e_o, A) - \{a, b\}$ as the union of two open arcs $C_1$ and $C_2$ where $C_1 - A \subset U$ and $C_2 - A \subset V$.**

If $C_2 - A = \phi$ there is nothing more to prove. If not the hypotheses implies that $C_2 \cap A = \phi$. In this case there are points in $V \cap E(A)$ that are close to $e_o$ and are not contained in the convex hull of $[a, e_o] \cup [e_o, b] \cup J$. If $e_1$ is such a point and $J_1 \in \mathcal{J}_A(e_1)$ then clearly $J_1 \subset T(J \cup A)$. This contradicts the hypotheses.

*Case II:* $J \notin \mathcal{J}_A(e)$ for any $e \in \overline{V}$. In this case choose $e_o \in U$ so that $J \in \mathcal{J}_A(e_o)$ and $d(e_o, J)$ is minimal. The proof now proceeds as in case I except that $e_1$ is chosen in the interior of the convex hull of $[a, e_o] \cup [e_o, b] \cup J$.

*THEOREM 4.2: Let $\overline{U}$ be an open set that contains A, let $\mathcal{J} = \{ J : J \in \mathcal{J}(A)$ and $J \subset U \}$, and let D be the boundary of $T(\cup\{J : J \in \mathcal{J}\} \cup A)$. Then D is a simple closed curve. If K is a bounded component of the complement of A and $e(K) \notin \overline{U}$ then the boundary of $T_K(\cup\{J: J \in \mathcal{J}\} \cup A)$, $D_K$ is likewise a simple closed curve.*

*Proof:* It will be proven here that $D$ is a simple closed curve. The proof for $D_K$ is similar.

Suppose it can be shown that for each $J = (a,b) \in \mathcal{J}(A)$, contained in the boundary of $CH(A)$, and not in $T(D)$, $CH(A)$ can be written as the union of two topological two cells $D(J)$ and $H(J)$ where $D \subset D(J)$ and $D(J) \cap H(J)$ is an arc $S(J) \subset D$, with endpoints $a$ and $b$. Since as a consequence of lemma 4.2 only finitely many of such $J$ exist, say $J_1, J_2, \ldots, J_n$ $T(D) = \cap\{D(J_i):$ $i = 1, 2, \ldots, n\}$. Since $A$ has no cutpoints it follows that the $S(J)$ are mutually disjoint and $T(D)$ is a two cell.

To see that this is indeed the case let $J_o = (a,b)$ as above. Let $\mathcal{J}'$ be the set of $J$ in $\mathcal{J}(A)$ that are not contained in the interior of $T(D)$, are contained in $T(J_o \cup A)$, and are not between $J_o$ and any $L$ in $\mathcal{J}(A)$ that is not contained in the interior of $T(D)$. Since $A$ is compact each $J \in \mathcal{J}'$ has diameter greater than or equal to some fixed $\delta > 0$. It follows from lemma 4.2 that $\mathcal{J}'$ is finite.

According to corollary 2 to theorem 4.1 $M(J_o) = \cup \{D(J_o, J) : J \in \mathcal{J}'\}$ is the interior of a closed two cell. Corollary 1 to theorem 4.1 states that there is a closed topological two cell $V$ such that $CH(A) = V \cup \overline{M(J_o)}$ and $V \cap \overline{M(J_o)}$ is an arc with endpoints $a$ and $b$. If $V \cap \overline{M(J_o)} \subset D$ there is nothing more to prove. If not a modification of the sets $V$ and $\overline{M(J_o)}$ can be obtained to do the job as follows: For $J \in \mathcal{J}'$ let $D'(J_o, J) = D(J_o, J)$ if $J \subset D$. If $J \not\subset D$, then $L \not\subset T(J \cup A)$ when $L \in \mathcal{J}(A)$. According to lemma 4.3 there is a circular arc in $A$ $Q(J)$ whose endpoints are the endpoints of $J$. Let $D'(J_o, J)$ be the interior of $\overline{T(D(J_o, J) \cup Q(J))}$. Let $M'(J_o) = \cup \{D'(J_o, J) : J \in \mathcal{J}'\}$, let $D(J_o)$ be the complement of $J_o \cup M'(J_o)$, and let $H(J_o)$ be the closure of $M'(J_o)$.

*LEMMA 4.4: Let $J \in \mathcal{J} \subset \mathcal{J}(A)$, let $K$ be a component of the complement of $A$ that contains $J$ and suppose $e(K) \not\in \overline{\cup \{J : J \in \mathcal{J}\}}$. Then there is a unique $L(J) \in \mathcal{J}(A)$ that satisfies: (i) If $J' \in \mathcal{J}$ and $J \subset T_K(J' \cup A)$ then $J' \subset T_K(L(J) \cup A)$, and (ii) If $L \in \mathcal{J}(A)$ satisfies (i) then $L(J) \subset T_K(L \cup A)$.*

*Proof:* The proof is straight forward and is left to the reader.

*THEOREM 4.3: Let $\mathcal{J} \subset \mathcal{J}(A)$, let $K$ be a component of the complement of $A$ and suppose $e(K) \notin \overline{\cup \{J : J \in \mathcal{J}\}}$. If $M = \cup \{J : J \in \mathcal{J}\} \cup A$ contains a simple closed curve $D$ such that $T_k(D) \supset A$ then the boundary of $T_K(M)$ is a* simple closed curve.

*Proof:* For each $J$ in $\mathcal{J}$ not contained in $T_K(D)$ let $L(J)$ be as in Lemma 4.4. Let $b$ be a homeomorphism of the unit circle $S^1$ onto $D$. If $L(J)$ has endpoints $a$ and $b$ let $S(J)$ be the component of $S^1 - \{b^{-1}(a), b^{-1}(b)\}$ for which $b(S(J))$ is contained in the interior of $T_K(L(J) \cup D)$. Define $b'$ on $S^1$ by $b'(z) = b(z)$ if $z$ is on no $S(J)$ and $b'$ maps each $S(J)$ homeomorphically onto $L(J)$. Lemma 4.2 can be applied to show that $b'$ is continuous.

*COROLLARY: If $A$ is a simple closed curve, $\mathcal{J} \subset \mathcal{J}(A)$ and $e(K) \not\in \overline{\cup \{J : J \in \mathcal{J}\}}$ then the boundary of $T_K(\cup\{J : J \in \mathcal{J}\} \cup A$ is a simple closed curve. Furthermore if $A$ has finite length then the length of the boundary of $T_K(\cup\{J : J \in \mathcal{J}\} \cup A)$ is less than or equal to the length of $A$.*

*THEOREM 4.4: Let $B$ be a compact set for which the interior of $T_K(B)$ contains $A$, let $\mathcal{J} = \{J : J \in \mathcal{J}_A(e)$ for some $e \in B\}$, and let $M = \cup\{J : J \in \mathcal{J}\} \cup A$. Then the boundary of $T_K(M)$ is a simple closed curve provided that $e(K) \not\in \overline{M}$.*

*Proof:* Since $A$ is compact there is a $\delta > 0$ for which $U_\delta = \{x : d(x,A) < \delta\}$ is contained in the interior of $T_K(B)$. Let $\mathcal{J}' = \{J : J \in \mathcal{J}(A)$ and $J \subset T_K(L \cup A)$ for some $L \in \mathcal{J}\}$. Let $M' = \cup\{J : J \in \mathcal{J}'\} \cup A$. Clearly $T_K(M') = T_K(M)$. By theorem 4.2 the boundary of $T_K(\cup\{J : J \in \mathcal{J}(A)$ and $J \subset U_\delta\} \cup A)$ is a simple closed curve that satisfies the hypothesis of theorem 4.3. Therefore the boundary of $T_K(M')$ is a simple closed curve.

*COROLLARY 4.4.2: Let $B$ be a compact set whose interior contains $A$. Then there is a continuum $M \subset B$ whose boundary consists of a finite number of mutually disjoint simple closed curves contained in $\cup\{J : J \in \mathcal{J}(A)\} \cup A$.*

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