ON MIKUSINSKI’S OPERATORS OF FRACTIONAL INTEGRATION

by

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ABSTRACT

In the field $F$ of convolution quotients, $b^\alpha$ is the operator of integration of fractional order $\alpha$ and $\hat{b}f$ is the Riemann-Liouville integral of order $\alpha$ of $f$. In this paper we give a generalization of this operator, which is denoted as $b^\alpha_{\hat{\alpha},\nu}$. Some particular cases are mentioned and the inverse operator is obtained.

1. Introduction. In a certain approach to the solution of mixed boundary value problems, an important part is played by certain operators of fractional integration. A brief account of operators of fractional integration is given in Sneddon’s book [13]. Operators of fractional integration involving generalized hypergeometric functions have been defined and discussed by Kalla [6, 7, 8]. Riemann-Liouville and Weyl fractional integrals and their connections with certain integral transforms are given in [3, 4, 5]. Kalla and Saxena [9] have discussed integral operators involving Gauss hypergeometric function $\mathbf{2}_F \mathbf{1}_F$, and they have established
their connections with the Hankel operator [10].

On the other hand, when we look into the field $F$ of convolution quotient $[1, 12]$, $b$, the constant function $[1]$, which is the restriction of Heaviside's unit function to the half line $t \geq 0$, plays an important role as an operator. $b^\alpha$ is regarded as the operator of integration of fractional order $\alpha$, and $b^\alpha f$ the Riemann-Liouville integral of order $\alpha$ of $f$.

The object of the present paper is to generalize the operator $b^\alpha f$. We denote the generalized operator as $b^\alpha f$. Several special cases of this operator are mentioned and the inverse operator is discussed. Integral operators involving Bessel functions can also be derived from our generalized operator.

2. Definition and Special Cases. We have [1, p. 141]

\begin{equation}
 b(t) = 1, \tag{1}
\end{equation}

\begin{equation}
 b(t) * b(t) = b^2(t) = \int_0^t b(x) b(t-x) \, dx = t, \tag{2}
\end{equation}

and

\begin{equation}
 b^\alpha = \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right), \quad \text{Re}(\alpha) > 0. \tag{3}
\end{equation}

Thus, the Riemann-Liouville fractional integration of order $\alpha$ may be considered as

\begin{equation}
 b^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt, \quad \text{Re}(\alpha) > 0. \tag{4}
\end{equation}

If we set

\begin{equation}
 f(t) = t^{\nu-1} e^{-at}, \quad \text{Re}(\nu) > 0, \quad \text{Re}(a) > 0, \tag{5}
\end{equation}

then [3, p.187]
\[ b^{\alpha}(s-a)^{-\nu} \leftrightarrow \frac{x^{\alpha + \nu - 1}}{\Gamma(\alpha + \nu)} \, \mathbb{I}^\nu_0 \left( \nu; \alpha + \nu; ax \right) , \]  

where

\[ b^{-1} = s \quad \text{and} \quad b^{-\frac{1}{2}} = s^{\frac{1}{2}} . \]  

The relation (6) can be rewritten as

\[ \frac{(s-a)^{-\nu}}{s^{\alpha}} \leftrightarrow \frac{x^{\alpha + \nu - 1}}{\Gamma(\alpha + \nu)} \, \mathbb{I}^\nu_0 \left( \nu; \alpha + \nu; ax \right) \]  

We shall denote the operator \( \frac{(s-a)^{-\nu}}{s^{\alpha}} \) by \( \bar{b}^{\alpha, \nu}_a \) thus

\[ \bar{b}^{\alpha, \nu}_a = \frac{(s-a)}{s^{\alpha}} \]  

For any elements \( g \in F \), we have

\[ \bar{b}^{\alpha, \nu}_a g = \frac{1}{\Gamma(\alpha + \nu)} \int_0^x (x-t)^{\alpha + \nu - 1} \mathbb{I}^\nu_0 \left( \nu; \alpha + \nu; a(x-t) \right) g(t) \, dt , \]

\[ \Re(\nu) > 0 , \quad \Re(\alpha) > 0 . \]  

When \( \nu \to 0 \)

\[ \bar{b}^{\alpha, 0}_a g \to b^{\alpha}_a g = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha - 1} g(t) \, dt , \quad \Re(\alpha) > 0 , \]

we obtain the Riemann-Liouville fractional integral of order \( \alpha \) (4).

We mention some special cases of our generalized operator \( \bar{b}^{\alpha, \nu}_a \).

(i) By virtue of the relation [11, p 271]

\[ \mathbb{I}^\nu_0 \left( \nu; \nu; ax \right) = e^{ax} \]  

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we have
\[ b_a^{\nu+1; \nu+\frac{1}{2}} g = \frac{1}{\Gamma(\nu+1)} \int_0^x (x-t)^{\nu-1} e^{a(x-t)} g(t) \, dt. \]  

Similarly on using the special cases of the confluent hypergeometric function \(_1 F_1\) [11, p. 271], we obtain the following particular cases of our operator:

(ii) \[ b_a^{\nu; \nu+\frac{1}{2}} g = \frac{2}{c \sqrt{\pi}} \int_0^x \text{Erf} \cdot [ c (x-t)^{\frac{1}{2}} ] g(t) \, dt, \]

where \( \text{Erf} \) stands for error function [11].

(iii) If we replace \( \alpha \) and \( \nu \) by \( \nu+1 \) and \( \nu+\frac{1}{2} \), respectively, then we obtain

\[ b_a^{\nu+1; \nu+\frac{1}{2}} g = \frac{2^{2\nu}}{\Gamma(\nu+1)} \int_0^x \exp \left( \frac{a(x-t)}{2} \right) I_{\nu+1} \left( \frac{a(x-t)}{2} \right) g(t) \, dt. \]

(iv) If \( \nu = \mu + \frac{1}{2} - k \) and \( \alpha = \mu + \frac{1}{2} + k \) then we get

\[ b_a^{\alpha+\frac{1}{2}+k; \mu+\frac{1}{2}-k} g = \frac{1}{\Gamma(\mu+1)} \int_0^x (x-t)^{-k} \exp \left( \frac{a(x-t)}{2} \right) \mathbf{M}_{k,\mu} \left[ a(x-t) \right] g(t) dt. \]

3. Inverse Operator. Theorem: If \( \text{Re}(\alpha) > 0 \), \( g \in F \), \( \varphi(0) = \varphi'(0) = 0 \), \( n > \text{Re}(\alpha+\nu) > 0 \) and

\[ b_a^{\alpha,\nu} g = \frac{1}{\Gamma(\alpha+\nu)} \int_0^x (x-t)^{\alpha-1} \varphi(1+\nu; a(x-t)) g(t) \, dt = \varphi, \text{(say)} \]

then

\[ g(x) = \frac{1}{\Gamma(n-\alpha+\nu)} \int_0^x (x-t)^{n-\alpha+\nu-1} \varphi(1-n+\nu; a(x-t)) \varphi(n) \, dt. \]

that is to say, if

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Proof: We have
\[ b_a^{\alpha, \nu} g = \varphi \]  
then
\[ g = b_a^{n-\alpha, -\nu}(n) \]  

\[ (s-a)^{-\nu} \]
\[ \frac{s^\alpha}{s} \]

hence
\[ g = s^\alpha (s-a)^{-\nu} \varphi \]

\[ = s^{-\nu-(n-\alpha-\nu)}(s-a)^{-\nu} (s^n \varphi) = b_a^{n-\alpha, -\nu}(\varphi(n)) \]  

by virtue of the result [1, p. 281],

\[ s^n f = f + \frac{n-1}{n} f'(0) + \frac{n-2}{n} f''(0) + \cdots + f^{(n)}(0) s^{n-1} \]  

The results (17) and (18) are in agreement with those given by Wimp [14].

References


6. S. L. Kalla: Fractional integration operators involving generalized hypergeo-


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