ON THE MIXTURE OF SOME CONTINUOUS DISTRIBUTIONS

by

G. S. LINGAPPAAIH

SUMMARY

In this note three aspects of the mixture of continuous distributions are considered. Firstly, a finite mixture of densities involving \( \phi(\frac{X_1 + \cdots + X_n}{\beta} ; \lambda_1 x, \ldots, \lambda_n x) \) is considered and the distribution of the sum of independent variables, each from such a mixture, is obtained. Secondly, characterization of a mixture of \( k \) Gamma distributions is attempted. Finally, non-central chi-square and doubly non-central \( F \) is discussed in relation to a finite mixture of normal distributions.

1. Introduction. Analysis of mixture of distributions has received attention for some time as can be seen in [17]. Many of the recent works on the mixture have been mainly on the problem of estimation of the parameters and the problem of identification which are illustrated in [13], [14], [15] and [16]. Various me-
methods, such as methods of moments [17] and the method of maximum likelihood in a grouped data [13] and [14] and other techniques are used for this purpose. Also, this author [4] has recently considered the distribution of the sum of independent variables, each with a finite mixture of Exponential distributions as their densities. In [21], for example, analysis of variance is discussed in relation to a mixture of two normal populations. Here, in this note, three aspects are dealt in relation to a finite mixture. Firstly, distribution of the sum of $s$ independent variables each from a finite mixture involving the hyper-geometric function [11]

$$
\phi_2 (\alpha_1, \alpha_2, \ldots, \alpha_n : \beta; \lambda_1 x, \ldots, \lambda_n x) \text{ is considered and it is shown that the results of [1] are the particular cases of this general situation. Incidentally, interesting applications of the General functions including hyper-geometric functions can be found in } [31], [5], [61], [7] \text{ and } [8].
$$

Secondly, a mixture of $k$-Gamma distributions is taken up and the characterization of such a mixture is attempted in terms of Inverted Dirichlet's distribution. Further it is shown that this is the general result of particular case of [9]. Finally a finite mixture of $K$-normal populations is considered and the non-central chi-square and doubly non-central $F$ distributions are dealt in relation to this mixture.

2. Distribution of the Sum. Consider a density

$$
f(x) = \sum_{j=1}^{k} \theta_j f_j(x), \quad 0 \leq \theta_j \leq 1, \quad \sum_{j=1}^{k} \theta_j = 1 \tag{1}
$$

where

$$
f_j(x) = \prod_{i=1}^{n} \left[ \left( \frac{\lambda_i}{x} \right)^{\alpha_{ij}} \right]^{-1} a^{a-1} \beta_j^{-1} a x^{a-1} \beta_j^j \text{ for } j = 1, 2, \ldots, k
$$

$$
\cdot \phi_2 (\alpha_1, \alpha_2, \ldots, \alpha_n : \beta; \lambda_1 x, \ldots, \lambda_n x) / \Gamma(\beta_j), \tag{2}
$$

$$
x > 0, \quad a > \lambda_1, \lambda_2, \ldots, \lambda_n; \quad a > 0, \quad \lambda_1, \ldots, \lambda_n \geq 0.
$$
and $\phi_2$ is the hyper-geometric function [11]. The characteristic function of $x$ using [10] is

$$\psi(t) = \sum_{j=1}^{k} \theta_j \psi_j(t),$$

where

$$\psi_j(t) = \prod_{i=1}^{l_j} \left[ \frac{(1 - \frac{\lambda_i}{a})}{\left(1 - \frac{\lambda_i}{a(1 - \frac{\theta_j}{a})}\right)} \right]^{\alpha_i} \cdot \frac{1}{\left(1 - \frac{\theta_j}{a}\right)^{\beta_j}}$$

Hence the characteristic function of the sum of $s$ independent variables each having (1) as its p.d.f., is

$$\psi^s(t) = \sum_{l_1, \ldots, l_k} \theta_1^{l_1} \cdots \theta_k^{l_k} \cdot \prod_{i=1}^{k} \left[ \frac{(1 - \frac{\lambda_i}{a})}{\left(1 - \frac{\lambda_i}{a(1 - \frac{\theta_j}{a})}\right)} \right]^{\alpha_i}$$

where $\alpha_i = \sum_{j=1}^{k} l_j \alpha_i$, $\beta = \sum_{j=1}^{k} l_j \beta_j$, and $\sum_{j=1}^{k} l_j = s$ and $\sum$ in (5) runs on the $l_j$'s. Inverting (5), we have the distribution of the sum as

$$f(u) = \sum_{l_1, \ldots, l_k} \theta_1^{l_1} \cdots \theta_k^{l_k} \cdot \prod_{i=1}^{k} \left[ \frac{(1 - \frac{\lambda_i}{a})}{\left(1 - \frac{\lambda_i}{a(1 - \frac{\theta_j}{a})}\right)} \right]^{\alpha_i} \cdot \phi_2 \left(\alpha_1, \alpha_2, \ldots, \alpha_n; \beta; \lambda_1 u, \ldots, \lambda_n u \right) / \Gamma(\beta).$$

where $u = \sum_{i=1}^{s} x_i$; now it is easy to see the result of $[11]$ is the special case of (6).

3. Mixture of Finite Number of Gamma distributions. Now, consider
\[
f(x) = \sum_{j=1}^{k} \theta_j \phi_j, \quad x > 0, \tag{7}
\]
where \( \phi_j(x) = e^{-x} \cdot x^{a_j - 1} / \Gamma(a_j) \) and the \( \theta_j \)'s are as before. We state below a characterization of (7) as follows.

**Theorem.** Necessary and sufficient condition that a set of independent random variables \( x_1, x_2, \ldots, x_{n+1} \) possess the density (7) is that a set of variables \( y_j = x_j / x_{n+1} \) \( j = 1, \ldots, n \), follow the distribution

\[
f(y_1, y_2, \ldots, y_n) = \sum_{j=1}^{l_1} \cdots \sum_{j=1}^{l_k} \frac{\theta_1 \cdots \theta_k}{\Gamma(a_1) \cdots \Gamma(a_n)} D(a_1, a_2, \ldots, a_n; a_{n+1}, j) \tag{8}
\]

where

\[
D(a_1, \ldots, a_n, a_{n+1}) = \frac{y_1^{a_1-1} \cdots y_n^{a_n-1} \Gamma(a_1, a_n+1)}{\left(1 + y_1 + \cdots + y_n\right)^{a+a+n+1} \Gamma(a_1) \cdots \Gamma(a_n) \Gamma(a_{n+1})} \tag{9}
\]

\( y_1, \ldots, y_n > 0 \) and \( a = a_1 + \cdots + a_n \).

The description of the two sums in (8) and the notation \( a_{ij} \)'s are described below.

**Necessity:** The joint density of \( x_1, \ldots, x_{n+1} \) is

\[
\prod_{i=1}^{n+1} f(x_i) = \prod_{i=1}^{n+1} \sum_{j=1}^{k} \theta_j \phi_{ij}(x) \tag{10}
\]

where

\[
\phi_{ij}(x) = e^{-x_i} \cdot x_i^{a_{ij}-1} / \Gamma(a_{ij}) \tag{10a}
\]

Now (10) can be written in the form

\[
\left| \sum_{j=1}^{l_1} \cdots \sum_{j=1}^{l_k} \phi_{ij_1} \phi_{ij_2} \cdots \phi_{ij_n} \right| \left| \sum_{s=1}^{k} \phi_{s(n+1), s} \right| \tag{10b}
\]
where \( r_j \)'s can take values 1, 2, \ldots, \( k \). Further the distribution of the \( r_j \)'s depend on \( l_j \)'s and the second sum in (10b). For example, if \( l_1 = n \), then all \( r_j \)'s are equal to 1. If \( l_k = n \), then all \( r_j \)'s are equal to \( k \). If \( n = 3 \) and \( k = 2 \), we have, for example, the coefficient of \( \theta_j^2 \theta_2 \) as

\[
(\phi_{11} \phi_{21} \phi_{32} + \phi_{12} \phi_{21} \phi_{31} + \phi_{11} \phi_{22} \phi_{31})
\]  

(10c)

Looking at (10c), we can notice that the first subscript of \( \phi \) corresponds to the observation number while the second subscript corresponds to the parameter within the observation. In general in (10b), \( l_1, r_j \)'s are 1, \( l_2, r_j \)'s are 2 and so on. Now if we make the transformation \( y_j = x_j / x_{n+1} \), \( j = 1, 2, \ldots, n \) and setting \( x_{n+1} = u \), we have the Jacobian of the transformation as \( u^n \) and (10b) reduces to

\[
f(y_1, y_2, \ldots, y_n; u) = \sum_{j=1}^{k} \theta_j \cdot \theta_k \cdot \frac{\sum_{j=1}^{n} y_j}{\prod_{j=1}^{k} (a_{l_j})} \cdot \frac{\sum_{j=1}^{n} y_j}{\prod_{j=1}^{k} (a_{r_j})}
\]

\[
\cdot \left[ \sum_{j=1}^{k} \theta_j e^{-u} a_{r_j+1, j-1} / \prod_{j=1}^{k} (a_{r_j+1, j}) \right]
\]

where \( a = a_{l_1} + \cdots + a_{r_n} \). Now integrating out \( u \) in (11), we get (8).

(ii) **Sufficiency:** To prove this part, we shall use the result of [12] according to which, if \( x_1, \ldots, x_{n+1} \) are independent, positive variables, then the characteristic function of \( \log y_1, \ldots, \log y_n \), determine the characteristic function of \( \log x_1, \ldots, \log x_{n+1} \), up to a term \( e^{\theta t b} \), provided the characteristic function of the latter do not vanish at any point. Hence, if the characteristic function of
\[ \log x_j \text{ is } \psi_j(t_j) \quad \text{then it is easy to see that} \]

\[
\psi(t_1, t_2, \ldots, t_n) = \psi_1(t_1) \cdots \psi_n(t_n) \psi_{n+1}(\sum_{j=1}^{n} t_j)
\]

(12)

where \( \psi(t_1, t_2, \ldots, t_n) \) is the characteristic function of \( \log y_1, \ldots, \log y_n \).

Now from (10a), we can see that

\[
\psi_j(t_j) = \sum_{s=1}^{k} \theta_s \left[ \frac{\Gamma(a_{js} + it_j)}{\Gamma(a_{js})} \right]
\]

(13)

and

\[
\psi_{n+1}(\sum_{j=1}^{n} t_j) = \sum_{s=1}^{k} \theta_s \left[ \frac{\Gamma(a_{n+1,s} + \sum_{j=1}^{n} i t_j)}{\Gamma(a_{n+1,s})} \right]
\]

(13a)

Now using (8), we get \( \psi(t_1, t_2, \ldots, t_n) \) as

\[
\sum_{l_1=1}^{l_1} \frac{1}{l_1} \cdots \frac{1}{l_k} \sum_{s=1}^{k} \frac{1}{l_s} \frac{\Gamma(a_{1r_{l_1}} + i t_{l_1}) \cdots \Gamma(a_{nr_{l_1}} + i t_{l_1}) \Gamma(a_{n+1,s} + \sum_{j=1}^{n} i t_j)}{\Gamma(a_{1r_{l_1}}) \cdots \Gamma(a_{nr_{l_1}}) \Gamma(a_{n+1,s})}
\]

(14)

This is exactly what we get for \( \psi_1(t_1) \cdots \psi_n(t_n) \psi_{n+1}(\sum t_j) \) using (13) and (13a) and hence the sufficiency follows. Note again that the second sum in (14) is on \( r_j \)'s which are controlled by \( l_j \)'s which in turn depend on the first sum.

Now if \( a_{fr_j} = a_j \), then in (8) every term has \( y_1 \cdots y_n \) and hence (8) reduces to (9) which is described in [9]. Further a comment on (8) is in order now. Actually, we have considered an intermediate situation. That is, we have taken \( \alpha = 1; \]

\[ e^{-\alpha x} x(\alpha x)^{a_{ij} - 1}/\Gamma(a_{ij}) \]

we could have let \( \alpha \) change instead \( a \) then (8) will be even simpler. In such an event a part of (8) will be

\[
\left[ \begin{array}{c} a_{1r_{l_1}}^{a_{ij} - 1} y_1^{a_{n+1,j} - 1} \\ \vdots \\ y_n^{a_{n+1,j} - 1} \end{array} \right] \cdot (a_{1r_{l_1}} \cdots a_{nr_{l_1}} a_{n+1,j})^d \cdot (\alpha_{n+1,j} + \alpha_{1r_{l_1}} y_1 + \cdots + \alpha_{nr_{l_1}} y_n)(n+1)a
\]

(15)
Similarly at the other end, we could have let both \( \theta \) and \( a \) change in which case part of (8) will look like

\[
\begin{pmatrix}
a_{1r_1} & \ldots & a_{nr_n} & a_{n+1,j} & a_{1r_1}^{-1} & a_{nr_n}^{-1} \\
a_{1r_1} & \ldots & a_{nr_n} & a_{n+1,j} & y_1 & \ldots & y_n
\end{pmatrix}
\]

(16)

where \( a = a_{1r_1} + \cdots + a_{nr_n} \)

4. Non-Central Chi-Square and doubly Non-Central - F in relation to the mixture. Now let

\[
f(x) = \sum_{j=1}^{k} \theta_j f_j(x)
\]

(17)

with same conditions as \( \theta_j \)'s as before and

\[
f_j(x) = \exp \left[ -\frac{1}{2} (x - \mu_j)^2 \right] \sqrt{2\pi} \quad -\infty < x < \infty.
\]

Then the joint density of \( n \)-independent variables is

\[
\prod_{i=1}^{n} \sum_{j=1}^{k} \theta_j f_{ij}(x_i)
\]

(18)

where \( f_{ij}(x_i) = \exp \left[ -\frac{1}{2} (x_i - \mu_{ij})^2 \right] \sqrt{2\pi} \) and (18) can be written as

\[
\sum_{\theta_1}^{l_1} \ldots \sum_{\theta_k}^{l_k} \sum_{i=1}^{l_1} \cdots \sum_{i=nr_n}^{l_n} f(x_1, \ldots, x_n)
\]

(19)

where again first subscript corresponds to the observation number and \( \theta_j \)'s take values 1, 2, \ldots, \( k \). Let \( u = \sum_{i=1}^{n} x_i^2 \), then the characteristic function of \( u \) is

\[
\phi(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) \exp(it \sum_{i=1}^{n} x_i^2) \, dx_1 \ldots \, dx_n
\]

(20)
which from (19) reduces to

\[
\psi(t) = \sum_{i} \theta_1^{\frac{1}{2}} \ldots \theta_k^{\frac{1}{2}} \sum_{s=0}^{\infty} e^{-\lambda \cdot \lambda^s / s!} (1 - 2it)^{-\left(\frac{n}{2} + s\right)}
\]  

(21)

where \( \lambda = (\mu_{11} + \ldots + \mu_{nn}) / 2 \). For example, the component \( f_{jr,j} \) gives

\[
\exp \left[-\frac{1}{2} (x_j - \mu_{jr,j})^2 + itx_j \right]
\]  

(21a)

on completing the square in (21a) and integrating out \( x_j \), we get

\[
\exp \left[-\frac{1}{2} \lambda_j^2 + \lambda_{j'}^2 / (1 - 2it) \right] / (1 - 2it)^{\frac{1}{2}}
\]  

where \( \lambda_j = \mu_{jr,j} \).

Inverting (21), we get

\[
f(u) = \sum_{i} \theta_1^{\frac{1}{2}} \ldots \theta_k^{\frac{1}{2}} \sum_{s=0}^{\infty} e^{-\lambda \cdot \lambda^s / s!} g(u) / s!
\]  

(22)

where \( \lambda \) is as above in (21) and

\[
g(u) = e^{-u/2} \left( \frac{u}{2} + s - 1 \right) / \Gamma \left( \frac{u}{2} + s \right)
\]  

(22a)

\( u > 0 \).

For example if \( n = 2, k = 2 \), we get

\[
f(u) = \left[ \sum_{s=0}^{\infty} \theta_1^{\frac{1}{2}} e^{-a_{11}} a_{11}^s + \theta_2^{\frac{1}{2}} \left[ e^{-a_{12}} a_{12}^s + e^{-a_{21}} a_{21}^s \right] + \theta_2^{\frac{1}{2}} e^{-a_{22}} a_{22}^s \right] g(u) / s!
\]  

(23)

where

\[
a_{11} = (\mu_{11} + \mu_{21}) / 2, \quad a_{21} = (\mu_{12} + \mu_{21}) / 2
\]
\[ a_{12} = \left( \frac{\mu_{11}^2 + \mu_{22}^2}{\mu_{11}^2 + \mu_{22}^2} \right) \cdot 2, \quad a_{22} \cdot \left( \frac{\mu_{12}^2 + \mu_{22}^2}{\mu_{12}^2 + \mu_{22}^2} \right) \cdot 2 \]

and
\[ g(u) = e^{-\frac{u}{2}} u^s \cdot \left( 1 + 2u \right)^{s+1} \]

It is to be noted again that in (22), the second sum runs over \( \lambda = \frac{\mu_{11}}{\mu_{11}} \cdot \ldots \cdot \frac{\mu_{n1}}{\mu_{n1}} \).

Now if we consider two independent variables \( v, u \), each having p.d.f. (22) and if we set \( F = \left[ (u/m) / v/n \right] \) and integrating out \( v \) in \( f(v) f(F \cdot v \cdot \frac{m}{n}) \), we get the resulting distribution as

\[ f(F) = \sum \frac{m!}{j_1 ! \ldots j_k !} \sum \frac{m_k}{j_k !} \exp \left[ -(\lambda + \delta) \right] \cdot \sum r \sum s \sum \lambda^s \delta^r g(F), F > 0 \quad (24) \]

where
\[ g(F) = \frac{m + n + s - 1}{B(\frac{m}{2} + s, \frac{n}{2} + r)} \quad (24a) \]

where in (24), first two sums are on \( j_i \)'s and \( m_j \)'s and second two sums are on \( \mu_j r_j \)'s and \( \delta_j r_j \)'s. (22) and (24) are respectively the mixture of non-central chi-square and doubly non-central \( F \) distributions. The non-centrality-parameter varies from term to term according as the second sum in (22). Further, probability integral from (24) is easy to compute using \( 181 \). Only thing new is the evaluation of the non-centrality parameter at each step.

REFERENCES


Department of Mathematics
Concordia University
Montreal, Canada

(Recibido en octubre de 1975).