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ON THE MIXTURE OF SOME CONTINUOUS DISTRIBUTIONS

by

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SUMMARY

In this note three aspects of the mixture of continuous distributions are considered. Firstly, a finite mixture of densities involving $\phi(\alpha_1, \alpha_2, \ldots, \alpha_n; \beta; \lambda_1 x, \ldots, \lambda_n x)$ is considered and the distribution of the sum of independent variables, each from such a mixture, is obtained. Secondly, characterization of a mixture of k Gamma distributions is attempted. Finally, non-central chi-square and doubly non-central F is discussed in relation to a finite mixture of normal distributions.

1. Introduction. Analysis of mixture of distributions has received attention for some time as can be seen in [17]. Many of the recent works on the mixture have been mainly on the problem of estimation of the parameters and the problem of identification which are illustrated in [13], [14], [15] and [16]. Various methods, such as methods of moments [17] and the method of maximum likelihood in a grouped data [13] and [14] and other techniques are used for this purpose. Also, this author [4] has recently considered the distribution of the sum of independent variables, each with a finite mixture of Exponential distributions as their densities. In [2], for example, analysis of variance is discussed in relation to a mixture of two normal populations. Here, in this note, three aspects are dealt in relation to a finite mixture. Firstly, distribution of the sum of s independent variables each from a finite mixture involving the hyper-geometric function [11] $\phi_2(\alpha_1, \alpha_2, \ldots, \alpha_n ; \beta ; \lambda_1 x, \ldots, \lambda_n x)$ is considered and it is shown that the results of [1] are the particular cases of this general situation. Incidentally, interesting applications of the General functions including hyper-geometric functions can be found in [3], [5], [6], [7] and [8]. Secondly, a mixture of k-Gamma distributions is taken up and the characterization of such a mixture is attempted in terms of Inverted Dirichlet's distribution. Further it is shown that this is the general result of particular case of [9]. Finally a finite mixture of K-normal populations is considered and the non-central chi-square and doubly non-central F distributions are dealt in relation to this mixture.

2. *Distribution of the Sum.* Consider a density

$$f(x) = \sum_{j=1}^{k} \theta_j f_j(x) \quad 0 \leq \theta_j \leq 1 , \sum_{j=1}^{k} \theta_j = 1 \tag{1}$$

where

$$f_j(x) = \prod_{i=1}^{n} \left[\left(1 - \frac{\lambda_i}{a} \right)^{\alpha_{ij}} \right] e^{-ax} x^{\beta_j - 1} a^{\beta_j}$$

$$\cdot \phi_2(\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{nj} ; \beta_j ; \lambda_1 x, \ldots, \lambda_n x)/\Gamma(\beta_j) , \tag{2}$$

$$x > 0 , a > \lambda_1, \lambda_2, \ldots, \lambda_n ; a > 0 , \lambda_1, \ldots, \lambda_n \geq 0 .$$

and ϕ_2 is the hyper-geometric function [11]. The characteristic fuction of x using [10] is

$$\psi(t) = \sum_{j=1}^{k} \theta_j \, \psi_j(t) \; . \tag{3}$$

where

$$\psi_j(t) = \prod_{i=1}^{n} \left[\left(1 - \frac{\lambda_i}{a}\right) \Big/ \left[1 - \frac{\lambda_i}{a\left(1 - \frac{\theta t}{a}\right)} \right] \right]^{\alpha_{ij}} \cdot \frac{1}{\left(1 - \frac{\theta t}{a}\right)^{\beta_j}} \tag{4}$$

Hence the characteristic function of the sum of s independent variables each having (1) as its p.d.f., is

$$\psi^s(t) = \sum \binom{s}{l_1, \ldots, l_k} \theta_1^{l_1} \cdots \theta_k^{l_k} \left(1 - \frac{\theta t}{a}\right)^{-\beta} \cdot \prod_{i=1}^{n} \left[\left(1 - \frac{\lambda_i}{a}\right) \Big/ 1 - \frac{\lambda_i}{a\left(1 - \frac{\theta t}{a}\right)} \right]^{\alpha_i} \tag{5}$$

where $\alpha_i = \sum_{j=1}^{k} l_j \alpha_{ij}$, $\beta = \sum_{j=1}^{k} l_j \beta_j$ and $\sum_{j=1}^{k} l_j = s$ and $\sum$ in (5) runs on the l_j's. Inverting (5), we have the distribution of the sum as

$$f(u) = \sum \binom{s}{l_1, \ldots, l_k} \theta_1^{l_1} \cdots \theta_k^{l_k} \prod_{i=1}^{n} \left(1 - \frac{\lambda_i}{a}\right)^{\alpha_i} a^{\beta}$$

$$\cdot\, e^{-au} \cdot u^{\beta - 1} \cdot \phi_2(\alpha_1, \alpha_2, \ldots, \alpha_n ; \beta ; \lambda_1 u, \ldots, \lambda_n u) / \Gamma(\beta) \; , \quad u > 0$$

where $u = \sum_{i=1}^{s} x_i$; now it is easy to see the result of [1] is the special case of (6).

3. *Mixture of Finite Number of Gamma distributions.* Now, consider

$$f(x) = \sum_{j=1}^{k} \theta_j \phi_j , \quad x > 0 , \tag{7}$$

where $\phi_j(x) = e^{-x} \cdot x^{a_j - 1} / \Gamma(a_j)$ and the θ_j's are as before. We state below a a characterization of (7) as follows.

THEOREM. Necessary and sufficient condition that a set of independent random variables $x_1, x_2, \ldots, x_{n+1}$ possess the density (7) is, that a set of variables $y_j = x_j / x_{n+1}$ $j = 1, \ldots, n$, follow the distribution

$$f(y_1, y_2, \ldots, y_n) = \sum \theta_1^{l_1} \ldots \theta_k^{l_k} \sum_{j=1}^{k} \theta_j D(a_{1r_1}, a_{2r_2}, \ldots, a_{nr_n} ; a_{n+1,j}) \tag{8}$$

where

$$D(a_1, \ldots, a_n ; a_{n+1}) = \frac{y_1^{a_1 - 1} \ldots y_n^{a_n - 1} \Gamma(a + a_{n+1})}{\left(1 + y_1 + \cdots + y_n\right)^{a + a_{n+1}} \cdot \Gamma(a_1) \ldots \Gamma(a_n) \Gamma(a_{n+1})} , \tag{9}$$

$$y_1, \ldots, y_n > 0 \quad \text{and} \quad a = a_1 + \cdots + a_n .$$

The description of the two sums in (8) and the notation a_{jr_j}'s are described below.

Necessity: The joint density of $x_1, \ldots, x_{n+1}$ is

$$\prod_{i=1}^{n+1} f(x_i) = \prod_{i=1}^{n+1} \sum_{j=1}^{k} (\theta_j \phi_{ij}(x)) \tag{10}$$

where

$$\phi_{ij}(x) = e^{-x_i} x_i^{a_{ij} - 1} / \Gamma(a_{ij}) \tag{10a}$$

Now (10) can be written in the form

$$[\sum \theta_1^{l_1} \ldots \theta_k^{l_k} \sum \phi_{1r_1} \phi_{2r_2} \cdots \phi_{nr_n}] [\sum_{s=1}^{k} \phi_{n+1,s}] \tag{10b}$$

where r_j's can take values $1, 2, \ldots, k$. Further the distribution of the r_j's depend on l_j's and the second sum in (10b). For example, if $l_1 = n$, then all r_j's are equal to 1. If $l_k = n$ then all r_j's are equal to k. If $n = 3$ and $k = 2$, we have, for example, the coefficient of $\theta_1^2 \theta_2$ as

$$(\phi_{11}\,\phi_{21}\,\phi_{32} + \phi_{12}\,\phi_{21}\,\phi_{31} + \phi_{11}\,\phi_{22}\,\phi_{31}) \tag{10c}$$

Looking at (10c), we can notice that the first subscript of ϕ corresponds to the observation number while the second subscript corresponds to the parameter within the observation. In general in (10b), l_1, r_j's are 1, l_2, r_j's are 2 and so on. Now if we make the transformation $y_j = x_j / x_{n+1}$, $j = 1, 2, \ldots, n$ and setting $x_{n+1} = u$, we have the Jacobian of the transformation as u^n and (10b) reduces to

$$f(y_1, y_2, \ldots, y_n ; u) = \Sigma\, \theta_1^{l_1} \ldots \theta_k^{l_k} \frac{e^{\sum_{j=1}^{n} u y_j}\, y_1^{a_{1r_1}-1} \ldots y_n^{a_{nr_n}-1}\, u^a}{\Gamma(a_{1r_1}) \ldots \Gamma(a_{nr_n})}$$

$$\cdot [\, \sum_{j=1}^{k} \theta_j\, e^{-u}\, u^{a_{n+1,j}-1} / \Gamma(a_{n+1,j}) \,] \tag{11}$$

where $a = a_{1r_1} + \ldots + a_{nr_n}$. Now integrating out u in (11), we get (8).

(ii) *Sufficiency :* To prove this part, we shall use the result of [12] according to which, if $x_1, \ldots, x_{n+1}$, are independent, positive variables, then the characteristic function of $\log y_1, \ldots, \log y_n$, determine the characteristic function of $\log x_1, \ldots, \log x_{n+1}$, up to a term $e^{\theta tb}$, provided the characteristic function of the latter do not vanish at any point. Hence, if the characteristic function of

$log\ x_j$ is $\psi_j(t_j)$ then it is easy to see that

$$\psi(t_1, t_2, \ldots, t_n) = \psi_1(t_1) \ldots \psi_n(t_n)\, \psi_{n+1}(-\sum_{j=1}^{n} t_j) \tag{12}$$

where $\psi(t_1, t_2, \ldots, t_n)$ is the characteristic function of $log\ y_1, \ldots, log\ y_n$.

Now from (10a), we can see that

$$\psi_j(t_j) = \sum_{s=1}^{k} \theta_s\, [\Gamma(a_{js} + it_j) / \Gamma(a_{js})] \tag{13}$$

and

$$\psi_{n+1}(-\sum_{j=1}^{n} t_j) = \sum_{s=1}^{k} \theta_s\, [\Gamma(a_{n+1,s} - \sum_{j=1}^{n} it_j) / \Gamma(a_{n+1,s})] \tag{13a}$$

Now using (8), we get $\phi(t_1, t_2, \ldots, t_n)$ as

$$\Sigma\, \theta_1^{l_1} \ldots \theta_k^{l_k}\, \Sigma \sum_{s=1}^{k} \theta_s \frac{\Gamma(a_{1r_1} + it_1) \ldots \Gamma(a_{nr_n} + it_n)\Gamma(a_{n+1,s} - \sum_{j=1}^{n} it_j)}{\Gamma(a_{1r_1})\Gamma(a_{2r_2}) \ldots \Gamma(a_{nr_n})\Gamma(a_{n+1,s})} \tag{14}$$

This is exactly what we get for $\psi_1(t_1) \ldots \psi_n(t_n)\psi_{n+1}(-\Sigma t_j)$ using (13) and (13 a) and hence the sufficiency follows. Note again that the second sum in (14) is on r_j's which are controlled by l_j's which in turn depend on the first sum. Now if $a_{jr_j} = a_j$, then in (8) every term has $y_1^{a_1 - 1} \ldots y_n^{a_n - 1}$ and hence (8) reduces to (9) which is described in [9]. Further a comment on (8) is in order now. Actually, we have considered an intermediate situation. That is, we have taken $\alpha = 1$; in $e^{-\alpha x} \alpha(\alpha x)^{a_{ij} - 1} / \Gamma(a_{ij})$ we could have let α change instead a Then (8) will be even simpler. In such an event a part of (8) will be

$$[y_1^{a_1 - 1} \ldots y_n^{a_n - 1}] \cdot \frac{(\alpha_{1r_1} \ldots \alpha_{nr_n} \cdot \alpha_{n+1,j})^a}{(\alpha_{n+1,j} + \alpha_{1r_1} y_1 + \ldots + \alpha_{nr_n} y_n)^{(n+1)a}} \tag{15}$$

Similarly at the other end, we could have let both α and a change in which case part of (8) will look like

$$\frac{\begin{pmatrix} a_{1r_1} & a_{nr_n} & a_{n+1,j} & a_{1r_1}-1 & a_{nr_n}-1 \\ \alpha_{1r_1} & \alpha_{nr_n} & \alpha_{n+1,j} & y_1 & \dots\ y_n \end{pmatrix}}{(\alpha_{n+1,j} + \alpha_{1r_1} y_1 + \dots + \alpha_{nr_n} y_n)^{a + a_{n+1,j}}} \tag{16}$$

where $a = a_{1r_1} + \dots + a_{nr_n}$

4. *Non-Central Chi-Square and doubly Non-Central - F in relation to the mixture.* Now let

$$f(x) = \sum_{j=1}^{k} \theta_j\, f_j(x) \tag{17}$$

with same conditions as θ' s as before and

$$f_j(x) = exp\,[-\tfrac{1}{2}(x-\mu_j)^2\,] / \sqrt{2\pi}\ ,\ -\infty \le x \le \infty\,.$$

Then the joint density of n-independent variables is

$$\prod_{i=1}^{n} \sum_{j=1}^{k} \theta_j\, f_{ij}(x_i) \quad , \tag{18}$$

where $f_{ij}(x_i) = exp\,[-\tfrac{1}{2}(x_i - \mu_{ij})^2\,] / \sqrt{2\pi}$ and (18) can be written as,

$$\Sigma\ \theta_1^{l_1} \dots \theta_k^{l_k}\ \Sigma\, f_{1r_1} \dots f_{nr_n} = f(x_1, \dots, x_n) \tag{19}$$

where again first subscript corresponds to the observation number and r_j's take values $1, 2, \dots, k$ Let $u = \sum_{i=1}^{n} x_i^2$, then the characteristic function of u is

$$\psi(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n)\, exp(it \sum_{i=1}^{n} x_i^2)\, dx_1 \dots dx_n \tag{20}$$

which from (19) reduces to

$$\psi(t) = \Sigma\, \theta_1^{l_1} \ldots \theta_k^{l_k}\, \Sigma\, (e^{-\lambda} \cdot \lambda^s / s!)\, (1 - 2it)^{-(\frac{n}{2} + s)} \tag{21}$$

where $\lambda = (\mu_{1r_1}^2 + \cdots + \mu_{nr_n}^2)/2$ For example, the component f_{jr_j} gives

$$exp\,[\, - \tfrac{1}{2}(x_j - \mu_{jr_j})^2 + itx_j^2\,] \tag{21a}$$

on completing the square in (21a) and integrating out x_j, we get

$$exp\,[\, - \tfrac{1}{2}\lambda_j^2 + \lambda_j^2 / (1 - 2it)\,] / (1 - 2it)^{\frac{1}{2}}$$

where $\lambda_j = \mu_{jr_j}$

Inverting (21), we get

$$f(u) = \Sigma\, \theta_1^{l_1} \ldots \theta_k^{l_k}\, \Sigma \sum_{s=o}^{\infty} e^{-\lambda} \cdot \lambda^s\, g(u) / s! \tag{22}$$

where λ is as above in (21) and

$$g(u) = e^{-u/2}\, u^{(\frac{n}{2} + s - 1)} / \Gamma(\tfrac{n}{2} + s)\, 2^{\frac{n}{2} + s} \tag{22a}$$

$$u > 0\,.$$

For example if $n = 2$, $k = 2$, we get

$$f(u) = \left[\sum_{s=o}^{\infty} \theta_1^2\, e^{-a_{11}}\, a_{11}^s + \theta_1 \theta_2 \left(e^{-a_{12}}\, a_{12}^s + e^{-a_{21}}\, a_{21}^s \right) + \theta_2^2\, e^{-a_{22}}\, a_{22}^s \right] g(u)/s! \tag{23}$$

where

$$a_{11} = (\mu_{11}^2 + \mu_{21}^2)/2, \qquad a_{21} = (\mu_{12}^2 + \mu_{21}^2)/2$$

$$a_{12} = (\mu_{11}^2 + \mu_{22}^2)/2, \qquad a_{22} = (\mu_{12}^2 + \mu_{22}^2)/2$$

and $g(u) = e^{-u/2} u^{s} / \Gamma(s+1) 2^{s+1}$

It is to be noted again that in (22), the second sum runs over $\lambda = \mu_{1r_1}^2 + \cdots + \mu_{mr_n}^2$. Now if we consider two independent variables v, u, each having p.d.f. (22) and if we set $F = [(u/m)/v/n)]^{1}$ and integrating out v in $f(v) f(F \cdot v \cdot \frac{m}{n})$; we get, the resulting distribution as

$$f(F) = \Sigma \Sigma\, \theta_1^{l_1+m_1} \cdots \theta_k^{l_k+m_k} \;\Sigma \Sigma\, exp[-(\lambda+\delta)] \cdot \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^s \delta^r}{s!\, r!}\, g(F),\ F>0 \quad (24)$$

where

$$g(F) = \frac{(m/n)^{\frac{m}{2}+s} F^{\frac{m}{2}+s-1}}{B(\frac{m}{2}+s, \frac{n}{2}+r)\,(1+\frac{m}{n}F)^{\frac{m+n}{2}+r+s}} \quad (24a)$$

where in (24), first two sums are on l_j's and m_j's and second two sums are on μ_{jr_j}'s and δ_{jr_j}'s. (22) and (24) are respectively the mixture of non-central chi-square and doubly non-central F distributions. The non-centrality-parameter varies from term to term according as the second sum in (22). Further, probability integral from (24) is easy to compute using [18]. Only thing new is the evaluation of the non-centrality parameter at each step

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