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# LOCALIZATION OF THE COHOMOLOGY OF A FINITE GALOIS GROUP IN A DEDEKIND DOMAIN

*by*

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Let us take a Dedekind domain $A$ with field of fractions $K$, $L$ a finite Galois extension of $K$ with Galois group $G$ and $B$ the integral closure of $A$ in $L$, then $B$ is a $G$-$A$-module and the cohomology groups of $G$ in $B$, denoted by $H^i(G,B)$ for $i$ any integer, are $A$-modules. Throughout this note we will denote by $Q$ (resp. by $P$) the set of non-zero prime ideals $q$ of $B$ (resp. $p$ of $A$). $\hat{L}_q$ (resp. $\hat{K}_p$), with $q \in Q$ (resp. $p \in P$), stands for the $q$-adic completion of $L$ (resp. $p$-adic completion of $K$) and $\hat{B}_q$ (resp. $\hat{A}_p$) is the corresponding ring of integers of $\hat{L}_q$ (resp. $\hat{K}_p$) ; also for $q \in Q$ lying over $p$ in $P$, $G_q$ will denote the decomposition group of $q$ in $L/K$ which is known to be Galois group of the Galois extension $\hat{L}_q/\hat{K}_p$. Our main aim is to prove :

*THEOREM : If $Q'$ is a subset of $Q$ containing exactly one divisor $q$ of each $p \in P$, then for any $i \in \mathbb{Z}$*

$$H^i(G,B) \cong \bigoplus H^i(G_q, \hat{B}_q) \quad (q \in Q').$$

First we will consider some auxiliary results.

*LEMMA 1 : If* $N^i$ *denotes de annihilator of* $H^i(G,B)$ *in* $A$ *(i any integer) and* $S: L \to K$ *is the trace, then for any* $i \in \mathbb{Z}$, $SB \subset N^i$ *; in particular,* $SB = N^0$.

*Proof:* It is clear that the multiplication in $B$ induces a cup product

$$\cup : H^i(G,B) \times H^j(G,B) \to H^{i+j}(G,B) \qquad (i,j \text{ in } Z);$$

from the properties of the cup product ([1], 4-1-9, 4-2-6), it follows that $H^0(G,B)$ is a ring and for any $i \in \mathbb{Z}$, $H^i(G,B)$ is a $H^0(G,B)$-module ; moreover, the isomorphism of groups $\bar{k}: A/SB \to H^0(G,B)$ induced by the epimorphism $k: A \to H^0(G,B)$ ([1], 2-2-6) is actually an isomorphism of rings.

Let us take now any $a \in A$ and any $\alpha \in H^i(G,B)$ represented by an $i$-cocycle $g$, then $a.\alpha$ is represented by $a.g$ and $ka \cup \alpha = a\,\alpha$ ([1], 4-3-6) ; in particular, if $a \in SB$ we get $a\alpha = 0$, i.e, $a \in N^i$.

*COROLLARY 1 : Suppose that* $K$ *is a local field, i.e* $K$ *is complete with respect to a discrete valuation. Then if* $L/K$ *is tamely ramified we have* $H^i(G,B) = 0$ *for any* $i \in \mathbb{Z}$.

*Proof:* $L/K$ tamely ramified implies $SB = A$ ([2], I-5 Thm 2) so, by Lemma 1, $N^i = A$ and therefore $H^i(G,B) = 0$ for any $i \in \mathbb{Z}$.

*COROLLARY 2 : For any* $i \in \mathbb{Z}$, $H^i(G_q, \hat{B}_q) = 0$ *for all but finitely many* $q \in Q$.

*Proof:* Given any $i \in \mathbb{Z}$ if $q \in Q$ is such that $H^i(G_q, \hat{B}_q) \neq 0$, then, by Corollary 1, $\hat{L}_q/\hat{K}_p$ is not tamely ramified so it can not be unramified either, and then $q$ divides the different $\mathfrak{D}_{L/K}$ ([3] ch. III, § 5) ; thus $q$ lies in the finite subset of $Q$ consisting of divisors of $\mathfrak{D}_{L/K}$

*PROPOSITION : If* $V_L$ *is the ring of restricted ádeles of* $B$ , $B_p = \amalg \hat{B}_q$ $L_p = \amalg \hat{L}_q$ , *where* $p \in P$ *and* $q \in Q$ *lie over* $P$, *then*

*(i)* $V_L$ , $B_p$ *and* $L_p$ *are G - module ;*

*(ii)* $H^i(G, V_L) = 0$ *for any* $i \in \mathbb{Z}$ .

*Proof :* (i) It is clear from the fact that given any $\sigma \in G$ it induces an iso - morphism

$$\sigma_q : \hat{L}_q \to \hat{L}_{\sigma_q} \quad \text{such that} \quad \sigma_q \hat{B}_q = \hat{B}_{\sigma_q}$$

(ii) $L_p$ is a vector space over $\hat{K}_p$ of dimension $n = [L : K]$ ; let us define the $\hat{K}_p$-linear map

$$S_{L_p/\hat{K}_p} : L_p \to \hat{K}_p \qquad \text{by} \qquad S_{L_p/\hat{K}_p}(x) = \Sigma\, S_{\hat{L}_q/\hat{K}_p}(x_q)$$

(for $q$ lying over $p$), where $x = (x_q) \in L_p$ and $S_{\hat{L}_q/\hat{K}_p}$ is the local trace . If $w_1, \ldots, w_n$ is a basis for $L/K$ we get a complementary basis $w_1^*, \ldots, w_n^*$ and since for $x \in L$ we have $S_{L_p/\hat{K}_p}(D_p(x)) = S_{L/K}(x)$ (see [2], ch 2 § 9), where $D_p$ is the diagonal imbedding of $L$ in $L_p$ , then

(1) $S_{L_p/\hat{K}_p}[D_p(w_i), D_p(w_j^*)] = S_{L/K}(w_i w_j^*) = \delta_{ij}$ $(i, j$ are integers between 1 and $n)$ ;

it follows that the $D_p(w_i)$'s are linearly independent over $\hat{K}_p$, i.e they form a basis for $L_p/\hat{K}_p$ . We define now a map

$$S_{V_L/V_K} : V_L \to V_K$$

by $(S_{V_L/V_K}(x))_p = S_{L_p/\hat{K}_p}(x_p)$, where $x = (x_p) \in V_L$ $(p \in P)$ and for each $p$, $x_p = (x_q) \in L_p$ ($q$ lying over $p$) ; then by (1) , each $x \in V_L$ can be written as

$$x = \sum_{i=1}^{n} S_{V_L/V_K}(x \cdot D(w_i^*)) \cdot D(w_i) ,$$

where $D$ is the diagonal imbedding of $L$ in $V_L$. Moreover, if $\sum_{i=1}^{n} a_i D(w_i) = 0$ with $a_i \in V_K$, then for any $i$, $a_i = 0$ ; in other words ,

(2) $$V_L = V_K D(w_1) \oplus \ldots \oplus V_K D(w_n)$$

finally, since $L/K$ is a finite Galois extension the basis $w_1, \ldots, w_n$ can be chosen to be normal and this, together with (2) and ([1], 3-1-3), completes the proof of (ii)

*LEMMA 2 : If $p \in P$, then for any $i \in \mathbb{Z}$, $H^i(G, B_p) = H^i(G_{q_0}, \hat{B}_{q_0})$ where $q_0$ is any fixed element in $Q$ lying over $p$. In particular, the cohomology groups $H^i(G_q, \hat{B}_q)$ for all $q$ lying over $p$ are canonically isomorphic .*

*Proof :* If we take for $G$ the coset decomposition $G = \bigcup \tau_i G_{q_0}$ $(1 \leq i \leq r)$ then $B_p = \prod \hat{B}_q$ ($q$ lying over $p$) $= \prod \hat{B}_{\tau_i q_0} = \prod \tau_i \hat{B}_{q_0}$ ; hence by Shapiro's Lemma, applied to $\{ G, G_{q_0}, B_p, \hat{B}_{q_0} \}$ (see [1], 3-7-15), the isomorphism follows.

*COROLLARY : For any $i \in \mathbb{Z}$, $H^i(G, L_p) = 0$ .*

*LEMMA 3 : If $V_B = \prod \hat{B}_q$ $(q \in Q)$ then*

*(i) For any $i \in \mathbb{Z}$, $H^i(G, V_B) = \oplus H^i(G_q, \hat{B}_q)$ $(q \in Q')$ where $Q'$ is a subset of $Q$ containing precisely one divisor $q$ of each $p \in P$ .*

*(ii) $V_B + D(L) = V_L$ .*

*Proof :* (i) Note that the direct sum makes sense because of Corollary 2, and since $V_B = \prod \hat{B}_q$ $(q \in Q) = \prod B_p$ $(p \in P)$ then (i) follows from Lemma 2 and the fact that the cohomology of finite groups commutes with direct products. (ii) follows from the Approximation Lemma ([3], ch. I § 3).

*Proof of the Theorem:* Let us consider the exact sequences of $G$-modules

$$0 \to L \xrightarrow{D} V_L \xrightarrow{\varphi} V_1/D(L) \to 0,$$

$$0 \to B \xrightarrow{D} V_B \xrightarrow{\varphi} V_B/D(B) \to 0.$$

If we look at the two induced long exact sequences of cohomology groups, and since $H^i(G,L) = 0 = H^i(G,V_L)$ for any $i \in \mathbb{Z}$, then $H^i(G, V_1/D(L)) = 0$. On the other hand

$$V_B/D(B) = V_B/V_B \cap D(L)$$

$$= V_B + D(L)/D(L) = V_L/D(L) \quad \text{(by Lemma 3 (ii))};$$

therefore

$$H^i(G, V_B/D(B)) = 0 \quad (\text{for } i \in \mathbb{Z}),$$

and so $H^i(G,B) = H^i(G,V_B)$. Lemma 3 (i) completes the proof. //

## REFERENCES


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