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A NOTE ON PERFECT MODULES OVER CROSSED PRODUCTS

by

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Let R be a ring with identity. We denote the Jacobson radical of R by $J(R)$ and assume throughout that $R/J(R)$ is Artinian.

We assume that G is a finite group of automorphisms of R that induces a completely outer group of automorphisms on $R/J(R)$. See Y. Miyashita [3, p. 126]. The crossed product Δ of R with G is $\sum_{\sigma \in G} \oplus Ru_\sigma$ with $(xu_\sigma)(yu_\tau) = xy^\sigma u_{\sigma\tau}$ for $x, y \in R$.

The fixed rings S is the set of $r \in R$ such that $r^\sigma = r$ for all $\sigma \in G$. In this way, R becomes a bi-Δ-S module. The Jacobson radical of Δ, R and S are denoted by $J(\Delta)$, $J(R)$ and $J(S)$ respectively.

Let M be a left Δ module; by $J({}_\Delta M)$, respectively $J({}_R M)$, we mean the radical of M, as a Δ module, respectively, as an R module.

Since $J(\Delta) = J(R)\Delta = \Delta J(R)$ by [5, Proposition 1, p. 187] we have

PROPOSITION 1. If P is a projective left Δ module, then $J({}_\Delta P) = J(\Delta) P = J(R) P = J({}_R P)$.

By $\ell.\dim_\Delta M$, we mean the projective dimension of M.

PROPOSITION 2. G acts as a Galois group for R if, and only if, $\ell.\dim_\Delta R < \infty$.

Proof: If G acts as a Galois group on R, then R is Δ projective. See [5, Proposition 2, p. 188] . Now Δ over R is a Frobenius extension. Hence by [4 , Theorem 8, p. 97] $\ell . dim_{\Delta} R = \ell . dim_{R} R = 0$. So by [5, Proposition 2, p. 188] G is a Galois group for R .

PROPOSITION 3. Let P and P' be projective left Δ modules. Then: a) If P is the R projective cover of $P/J(R)P$ and there is an R monomorphism from $P/J(R)P$ to $P'/J(R)P'$, then there is a Δ monomorphism from P to P' which splits. b) If P' is the projective cover of $P'/J(R)P'$ and there is an R epimorphism from $P/J(R)P$ to $P'/J(R)P'$, then there is a Δ epimorphism from P to P' which splits. c) If either P is the projective cover of $P/J(R)P$ or P' is the projective cover of $P'/J(R)P'$ and $P/J(R)P$ is R isomorphic to $P'/J(R)P'$, then P and P' are Δ isomorphic.

Proof: Assume M is a completely reducible left Δ module. Now $R/J(R) = U_1 + \dots + U_k$, where the $U_i's$ are minimal G invariant two sided ideals of R . Thus $\Delta/J(\Delta) = \Delta(U_1 : G) + \cdots + \Delta(U_k : G)$, where $\Delta(U_i : G)$ is the crossed product of U_i and G .

Let r_i be the number of isomorphic irreducible Δ components of $\Delta(U_i:G)M$. Each such component is Δ isomorphic to an irreducible left ideal of $\Delta/J(\Delta)$. Now each irreducible left ideal of Δ in $\Delta(U_i:G)$ is a direct sum of R modules, which are decomposed into a_i irreducible R modules . The numbers $r_i a_i$ of mutually isomorphic R irreducible components of M are determined by the R-structure. From $r_i a_i, r_i$ can be found. Hence the R-structure of M determines the Δ structure.

If N is a completely reducible Δ module, which is the R epimorphic M, then by the above M is Δ epimorphic to N .

We now show b). Since $P/J(R)P$ and $P'/J(R)P'$ are completely reducible left Δ modules, we can find a Δ epimorphism f from $P/J(R)P$ to $P'J(R)P'$. Let $\pi : P \to P/J(R)P$ and $\pi' : P' \to P'/J(R)P'$ be the natural maps. Since P. is Δ projective, we can find a Δ map g from P to P' such that $\pi' g = f\pi$. Now P' is the projective cover of $P'/J(R)P$, hence $J(R)P'$ is R small in P' . See G. Azumaya [1, Proposition 4] . Thus g is a split epimorphism.

Proof of c). Assume π' is a minimal R epimorphism ; hence minimal Δ epimorphism. It follows from the above that there is a Δ isomorphism f from $P/J(R)P$ to $P'/J(R)P'$. f can be lifted to a Δ epimorphism $g: P \to P'$. Also f^{-1} can be lifted to h. Thus $hg(x) - x \in \ker \pi = J(\Delta)P$. By [(2, Theorem 3, p. 233] hg is an injection, hence g is a injection.

Proof of a). Assume there is an R monomorphism from $P/J(R)P$ to $P'/J(R)P'$, let f be the Δ monomorphism from $P/J(R)P$ to $P'/J(R)P'$. Let $\pi: P \to P/J(R)P$ and $\pi': P' \to P'/J(R)P'$ be the natural maps. Since P is projective there is a Δ map $g: P \to P'$ such that $\pi' g = f\pi$. Let $\bar{g}$ be the obvious map from $P/J(R)P$ to $P'/J(R)P'$. Then $\bar{g} = f$. Since $\Delta/J(\Delta)$ is artinian f splits, say hf = the identity on $P/J(R)P$. Let k be a left Δ map from P' to P such that $\bar{k} = h$. Now $\overline{hg} = \bar{h}\bar{g}$ is the identity on $P/J(R)P$. Since P is the projective cover of $P/J(R)P$ we can use the above proof of c) to conclude hg; is the identity on P.

A projective module P is called semiperfect if every homomorphic image of P has a projective cover, while P is perfect if every (infinite) direct sum of copies of P is semiperfect. See G. Azumaya [1] . Now E. A. Rutter and R. S. Cunningham in [7] have shown P is perfect if and only if $P/J(P)$ is semisimple and $J(I)$ is left T nilpotent, where I is the trace ideal and $J(I)$ is the Jacobson radical of I as a submodule (ie. not as a ring).

Obviously, Proposition 3 can be used if P or P' is a semiperfect left Δ module.

PROPOSITION 4. Assume P is a projective left Δ module such that P, as an R module is perfect. Then P as a Δ module if perfect.

Proof: Since P is Δ projective $J({}_\Delta P) = PJ(\Delta)P = J(R)P = J({}_R P)$. Thus $P/J({}_\Delta P) = P/J(R)P$ which is completely reducible as an R module, hence as a Δ module. See [5, p. 188] . We will denote by $tr_\Delta P$ (resp. $tr_R P$) the trace of P as a Δ module (resp. the trace as an R module). Of course, $J(tr_\Delta P)$ (resp. $J(tr_R P)$) will denote the Jacobson radical of $tr_\Delta P$ as a Δ submodule (resp. as an R submodule). Since P is Δ projective $tr_\Delta P \cdot P = P$. So $J({}_\Delta P) = J(\Delta) \cdot P = J(\Delta) \cdot tr_\Delta P \cdot P$. Now $tr_\Delta P$ is a homomorphic image of a direct sum of copies of P. In fact, $tr_\Delta P/J(\Delta) \cdot tr_\Delta P$ is a homomorphic image of $P/J(\Delta)P$; hence $tr_\Delta P/J(\Delta) \cdot tr_\Delta P$ is semisimple. Thus $J(tr_\Delta P) = J(\Delta)\, tr_\Delta P$. We now show $J(tr_\Delta P)$ is T nilpotent. Since $J(tr_\Delta P)$ is a two-sided ideal of Δ, $tr_\Delta P=(tr_\Delta P\cap R)\Delta=\Delta\cdot(tr_\Delta P\cap R)$. Thus $J(tr_\Delta P) = J(\Delta)\cdot tr_\Delta P = J(R)\cdot tr_\Delta P = J(R)\cdot(tr_\Delta P\cap R)\cdot\Delta\subseteq J(R)\,tr_R P\cdot\Delta\subseteq J(tr_R P)\cdot\Delta$. Now $J(tr_R P)\cdot\Delta$ is left T nilpotent. Take any left Δ module M, then $J(tr_R P)\Delta\cdot M = J(tr_R P)\cdot M \neq M$, since $J(tr_R P)$ is left T nilpotent. Thus P is Δ perfect.

COROLLARY. If R is left perfect, Δ is left perfect.

PROPOSITION 5. Let M be a projective left Δ module and N a projective Δ submodule of M. Assume $N/J(N)N$ has a projective cover, as an R module. If N, as an R module, is a direct summand of M, then N is a Δ direct summand of M.

Proof. Follows from Proposition 3b).

COROLLARY. Let N be a projective Δ submodule of a projective Δ module M. Then if N, as an R-module, is semiperfect and a direct summand of M, then N is a Δ direct summand of M.

COROLLARY. Let R be a left perfect ring. If every finitely generated projective submodule of a projective R module is an R direct summand, then every finitely generated projective Δ submodule Q of a projective Δ module P is a direct summand

Proof. Now P is R projective and Q is a finitely generated projective R module; hence Q is an R direct summand.

Since Q is finitely generated, Q is the projective cover of $Q/J(R)Q$. Thus Q is a Δ direct summand.

By [2, Theorem 5.4, p. 480] we have: if the left annihilator of a finitely generated proper right ideal of R is always nonzero, then the left annihilator of a finitely generated proper right ideal of Δ is always nonzero.

Let P be a left Δ module. Assume P as an R module is projective and $P/J({}_R P)$ has a projective cover, as an R module. By $J({}_\Delta P)$ (resp. $J({}_R P)$) we mean the Jacobson radical of P as a Δ module (resp. as an R module).

PROPOSITION 6. a) A left Δ submodule Y of P is small as a Δ module if *and only if Y is small as an R module.*

b) If P has a projective cover as a Δ module, then P is Δ projective.

Proof of a). Since $\Delta/J(\Delta) = \Delta/J(R)\Delta$ is Artinian, $J({}_{\Delta}P) = J(\Delta)P = J(R)P = J({}_{R}P)$. Now $J({}_{R}P)$ is R small in P; hence $J({}_{\Delta}P)$ is Δ small in P. See [1, Proposition 4]. We assume Y is Δ small, hence Y is contained in every maximal left Δ module. Thus $Y \subseteq J({}_{\Delta}P)$, which is R small.

Proof of b). Let $f: Q \to P \to 0$ be the Δ cover of P. Then the kernel of f is Δ small, hence by 1) ker f is R small. Since P is R projective, f splits. Thus ker $f=0$. Hence P is Δ projective.

COROLLARY. Every left Δ module P which has a Δ cover and when viewed as an R module is semiperfect, is Δ projective.

References

1. G. Azumaya, *Characterizations of semiperfect and perfect modules*, to appear.
2. I. Beck, *Projective and free modules*, Math. Z. 129(1972), p. 231-234.
3. Y. Miyashita, *Finite outer Galois theory of non-commutative rings*, J. Fac.Sci. Hokkaido Univ. Ser. I 19(1966), pp.114-134.
4. T. Nakayama and T. Tsuzuku, *On Frobenius Extensions I*, Nagoya Math. J. 19 (1961), p. 127-148.
5. J. Osterburg, *Completely outer groups of automorphisms acting on $R/J(R)$*, Proc. Amer. Math. Soc. 46(1974), pp. 187-190.
6. ________, *Completely outer Galois theory of perfect rings*, Pac. J. of Math. 56(1975).
7. E. A. Rutter and R. S. Cunningham, *Perfect Modules*, to appear.

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