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ON THE ERGODIC THEORY OF CONTRACTIONS

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SUMMARY

We give a formulation of the ergodic theory of Banach space contractions which has as a special case Sine's finite dimension criterion for C(X). For Grothendieck spaces, a sharper condition for ergodicity of an operator is given, and a known mean divergence theorem for G-spaces of type C(X) is shown to hold (in suitable form) for any G-space. Finally we show that for G-spaces ergodicity of T is closely related to that of the adjoint T^*.

§ 1. *Criteria for mean convergence.* **Throughout, B will be a Banach space, T a linear operator on B with $||T|| \leq 1$, B^* the dual space of B, and T^* the adjoint of T. Some special notations are : $A_n(T) = (1/n)(T + \cdots + T^n)$, $F(T) = \{x \in B: Tx = x\}$, and $F(T^*) = \{m \in B^* : T^*m = m\}$. This paper is concerned with conditions under which T is strongly ergodic, i.e., there exists a projection P such that $|| A_n(T)x - Px || \to 0$ for all $x \in B$. Our formulation of ergodic theory is based on the known results quoted in 1.1.**

1.1. LEMMA. (a) $[S_1]$ T is strongly ergodic iff $F(T)$ separates points of $F(T^*)$, i.e., if m and n are distinct elements of $F(T^*)$, then $m(x) \neq n(x)$ for some $x \in F(T)$.

(b) [L1] *For any contraction T on B, there exists a projection Q^* on B^* and a net $\{A_{n(a)}(T^*) : a \in A\}$ such that $A_{n(a)}(T^*)m(x) \to Q^* m(x)$ for all $m \in B^*$ and $x \in B$. Range $Q^* = F(T^*)$, and $T^* Q^* = Q^* T^* = Q^*$. Moreover, the compactness argument of the proof yields the following: if $\{A_{n(a)}(T) : a \in A\}$ is any net with $n(a) \to \infty$, then there exists a subnet $\{A_{n(b)}(T^*) : b \in B\}$ converging to a projection Q^* as above.*

1.2. Remark. If Q^* is as above, then range $Q^* = F(T^*)$, and it is easy to see that $ker\ Q^*$ is a norm closed subspace of B^* containing $(I-T)^*(B^*)$. More generally, we have

$$(*)\ \text{norm-closure}\ (I-T)^*(B^*) \subset ker\ Q^* \subset \text{weak-* closure}\ (I-T)^*(B^*) = F(T)^{\perp}.$$

Note that if T is strongly ergodic, then Q^* is given uniquely as P^*, where P is a projection on B, and $ker\ Q^* = F(T)^{\perp}$, by weak-* continuity of Q^*.

1.3. PROPOSITION. (a) $F(T)^*$ is a homomorphic image of $F(T^*)$, (b) the homomorphism is an isomorphism iff T is strongly ergodic.

Proof. (a) Since $F(T)^*$ is isomorphic to $B^*/F(T)^{\perp}$ [R_1, page 91], we can deal with the quotient space. Let $\pi : B^* \to B^*/F(T)^{\perp}$ be the natural map. We shall show that the restriction $\pi | F(T^*)$ is onto, i.e., for each $m \in B^*$ there exists $n \in F(T^*)$ with $m-n \in F(T)^{\perp}$. In fact, let $n = Q^* m$. Then if $x \in F(T)$,

$$(m-n)(x) = m(x) - \lim A_{n(a)}(T^*)m(x) = m(x) - \lim m(A_{n(a)}(T)(x)) = m(x) - m(x) = 0.$$

(b) By 1.1 (a), T fails to be strongly ergodic iff there exist m and n in $F(T^*)$ with $m-n \in F(T)^{\perp}$, or $\pi(m) = \pi(n)$.

1.4. COROLLARY. If $\dim F(T^*) < \infty$, then $\dim F(T) \leq \dim F(T^*)$, and equality holds iff T is strongly ergodic.

(For the case $B = C(X)$, this result is due to Sine, [S_2] and [S_3]. He also relates ergodicity to the number of extreme points in the unit ball of $F(T^*)$.)

§ 2. *Grothendieck spaces.* In [Li], Lin proves that if $n^{-1} \| T^n \| \to 0$, then $\| A_n(T) - P \| \to 0$, for some projection P iff $(I-T)(B)$ is closed in B. By Banach's theorem [R_1, page 96], this last condition is equivalent to either of the assertions: $(I-T)^*(B^*)$ is norm-closed, or is weak-*closed. In view of 1.2, we may expect an analogous result for strong ergodicity, if only we can locate $ker\ Q^*$ with more precision. This turns out to be possible if B is a Grothendieck space. (B is a G-space if weak-* sequential convergence in B^* is equivalent to weak sequential convergence. A reflexive space is a G-space, and $C(X)$ is a G-space whenever X is a compact F-space [see, Theorem 2.5].)

2.1. LEMMA. If B is a G-space, then norm-closure $(I-T)^*(B^*) = \bigcap\{ker\ Q^*: Q^* \in S\}$, where S is the set of projections on B^* given as in 1.1 (b).

Proof. Suppose $m \in B^*$ and $m \notin$ norm-closure $(I-T)^*(B^*)$. By Hahn-Banach there exists $F \in B^{**}$ such that $F(m) = 1$ and $F(n - T^*n) = 0$ for all $n \in B^*$, or $F(n) = F(T^*n)$ for all $n \in B^*$. This implies that it is not true that $A_n(T^*)m \to 0$ weakly, because $F(A_n(T^*)m) = F(m) = 1$. Since B is a G-space, it is not true that $A_n(T^*)\,m \to 0$ weak-*, and hence there exist $x \in B$, $\epsilon > 0$, and $n(1) < n(2) < \cdots$ such that $|A_{n(k)}(T^*)\,m(x)| > \epsilon$. By 1.1 (b) there exists a subnet $\{A_{n(a)}(T^*) : a \in A\}$ of the sequence $\{A_{n(k)}(T^*)\}$ converging in the specified sense to some $Q^* \in S$. Clearly $|Q^*m(x)| \geq \epsilon$, so $m \notin ker\ Q^*$.

2.2. THEOREM. If B is a G-space, then T is strongly ergodic iff norm-closure $(I-T)^*(B^*) =$ weak- closure $(I-T)^*(B^*)$.*

Proof. If T is strongly ergodic, then (as noted in 1.2) Q^* is given uniquely, and $ker\ Q^* =$ weak-* closure. By 2.1 it follows that norm-closure = weak-* closure. Conversely, if norm-closure = weak-* closure, then any $Q^* \in S$ must satisfy $ker\ Q^* = F(T)^{\perp}$, and range $Q^* = F(T^*)$; whence $F(T)^{\perp} \cap F(T^*) = (0)$. But this set is the kernel of the map $\pi : F(T^*) \to B^*/F(T)^{\perp}$, so that 1.3 (b) implies that T is strongly ergodic.

2.3. Examples. These results can fail for non-G spaces. Before giving examples, we need a Lemma, the proof of which was kindly supplied by Michael Lin.

LEMMA. Let X be compact, $S: X \to X$ a homeomorphism such that for some $x \in X$, the orbit $\{S^n x\}$ is infinite, and T the Markov operator $T f(x) = f(Sx)$ ($f \in C(X)$). Then there exist x and y in X such that

$$\delta_x - \delta_y \in F(T)^{\perp} \setminus \text{norm-closure } (I-T)^*(C(X)^*).$$

Proof. If $\{S^n x\}$ is infinite, then by compactness there exists $y \in$ closure $\{S^n x\}$ such that $\{S^n x\} \cap \{S^n y\} = \phi$. Since each element of $F(T)$ is constant on closure $\{S^n x\}$, we have $\delta_x - \delta_y \in F(T)^\perp$. Now for each n, choose $f_n \in C(X)$ such that $\|f_n\| \leq 1$, $f_n(S^i x) = 1$ $(i = 1, \ldots, n)$, and $f_n(S^i y) = -1$ $(i = 1, \ldots, n)$. Then $A_n(T^*)(\delta_x - \delta_y) f_n = 2$, so $\|A_n(T^*)(\delta_x - \delta_y)\| \not\to 0$. It follows that $\delta_x - \delta_y \notin$ norm-closure $(I - T)^*(C(X)^*)$.

Now for our first example (from $[S_3]$), let $X = [0,1]$, $Sx = x^2$, and $Tf(x) = f(Sx)$. T is not strongly ergodic, since $T^n f(x) \to f(0)$ if $x \neq 1$, and $T^n f(1) \to f(1)$. However the sequence $\{T^{*n}\}$ is convergent in the sense of 1.1 (b), since $T^{*n} m \to m[0,1)\delta_o + m\{1\}\delta_1$ for each $m \in C(X)^*$ Thus Q^* is unique, and $m \in ker\, Q^*$ iff $m[0,1) = m\{1\} = 0$. Clearly, $\delta_o - \delta_1 \in F(T)^\perp \setminus ker\, Q^*$. If $0 < x < y < 1$ and $\{S^n x\}$ is disjoint from $\{S^n y\}$, then the Lemma implies $\delta_x - \delta_y \in ker\, Q^* \setminus$ norm-closure $(I-T)^*(C(X)^*)$ Thus 2.1 fails.

For a second example, let X be the unit circle, $S: X \to X$ the map $Sz = az$, where a is not a root of unity, and $Tf(z) = f(Sz)$. In this case, T is strongly ergodic, with $A_n(T)\, f(x) \to \int f\, dm$, where m is normalized Lebesgue measure, and hence $ker\, Q^* = F(T)^\perp$. Since $F(T)$ = constant functions, $F(T)^\perp = \{n \in C(X)^* : n(X) = 0\}$, and by the Lemma there exist x and y such that $\delta_x - \delta_y \in F(T)^\perp \setminus$ norm-closure. Thus, 2.1 and 2.2 both fail.

2.4. Remark. If S and T are as in the Lemma, and $C(X)$ is a G-space, then it follows easily form 2.2 and the Lemma that T is not strongly ergodic, an essentially known result [Sem]. However, we can prove a stronger result.

2.5. DEFINITION. If B is a Banach space, a sequence $\{\rho_n\} \subset B^*$ is said to have *disjoint supports* if $\|\Sigma_1^R t_n \rho_n\| = \Sigma_1^R |t_n| \cdot \|\rho_n\|$ for any finite set of scalars $\{t_1, \ldots, t_R\}$.

2.6. THEOREM. Suppose T is a contraction on a G-space B such that for some $\rho \in B^*$, $\{T^{*n}\rho\}$ have disjoint supports, and $\inf \|T^{*n}\rho\| > 0$. Then T is not strongly ergodic.

This Theorem follows from the following Lemma. For point functionals in a compact F-space, the Lemma was proved in $[R_2]$ and $[G\text{-}K]$, and more generally for compact spaces such that $C(X)$ is a G-space in [Sem]. (I owe this last reference to W.D. Stangl, who proved in his 1974 Lehigh thesis the more general result that such a compact space contains no 'heavy points' in the sense of A. K. Snyder.)

2.7. LEMMA. Let B a G-space, $\{\rho_n\} \subset B^*$ with disjoint supports, $\|\rho_n\| \leq M$, and $\inf \|\rho_n\| > 0$. Then there exists $x \in B$ such that the sequence $\{\rho_n(x)\}$ is not Cèsaro summable.

Proof. Let $\{c_n\}$ be a bounded real sequence which is not Cèsaro summable. It follows from page 86 of [D-S] that there exists $F \in B^{**}$ such that $F(\rho_n) = c_n$ for all n. Let $m_n = (1/n)(\rho_1 + \cdots + \rho_n)$. Then $\{F(m_n)\}$ is a divergent sequence, so that $\{m_n\}$ is not a weakly convergent sequence in B^*. Since B is a G-space, the sequence is not weak-* convergent, so there exists $x \in B$ such that $\{m_n(x)\}$ diverges, i.e., $\{\rho_n(x)\}$ is not Cèsaro summable. (We note that in these results Cèsaro summability may be replaced by any regular matrix method.)

3. Ergodicity of T^*. If T is strongly ergodic, then T^* is weak-* ergodic, but may not be strongly ergodic. (For example, let $B = C[0,1/2]$ and $Tf(x) = f(x^2)$.) In G-spaces we can again do better.

3.1. THEOREM. Let T be a contraction on the G-space B. Then (a) implies (b) implies (c), where

(a) T is strongly ergodic,
(b) T^* is weak- ergodic,*
(c) T^* is strongly ergodic.

Proof. That (a) implies (b) is obvious. Assume (b) holds, i.e., there exists a projection P on B^* such that for each $\rho \in B^*$, $A_n(T^*)\rho \to P\rho$ weak-*. To prove T^* is strongly ergodic, it is enough to show that if $G \in F(T^{**}) \cap F(T^*)^{\perp}$, then $G=0$. (Cf. 1.3 and the proof of 2.2.) If $G \neq 0$, then there exists $\rho \in B^*$ with $G(\rho) \neq 0$. Then $A_n(T^*)\rho \to P\rho$ weak-*, where $P\rho \in F(T^*)$. Since B is a G-space, convergence is weak as well, so $0 \neq G(\rho) = A_n(T^{**})G(\rho) = G(A_n(T^*)\rho) \to G(P\rho) = 0$, since $G \in F(T^*)^{\perp}$ and $P\rho \in F(T^*)$. Thus we have a contradiction.

3.2. COROLLARY. Let T be a contradiction on a Banach space B, and assume that B^* is a G-space. Then (c) implies (a).

Proof. If T^* is strongly ergodic, then 3.1 implies that T^{**} is strongly ergodic. Since B is a norm-closed subspace of B^{**}, it follows that T is strongly ergodic.

3.3. Example. To show that the hypothesis in 3.2 is really needed, we give an example where T^* is strongly ergodic while T is not. Define T on c_o (= null sequences) by $(Tx)_1 = x_1$ and $(Tx)_n = x_{n-1}$ for $n > 1$. For each $x \in c_o$, the sequence $\{ (T^k x) : k \geq 1 \}$ converges to $(x_1, x_1, x_1, \ldots)$, which is not in c_o if $x_1 \neq 0$. It is easy to check that the adjoint $S = T^*$ is given for $y \in \ell^1$ by

$$Sy = (y_1 + y_2, y_3, y_4, \ldots).$$

The iterates $S^k y$ converge pointwise and in ℓ^1-norm to $(\Sigma_i y_i, 0, 0, \ldots)$. (Note that the projection so defined is norm but not weak-* continuous on ℓ^1.)

It is apparently not known whether there exist non-reflexive spaces such that both B and B^* are G-spaces [*D*, page 105], so it is not clear whether 3.1 and 3.2 have non-trivial joint applications.

4. Final Remark. Corollary 1.4 has been proved independently by Michael Lin, for a semigroup of contractions. His proof embeds the π-invariant vectors in the dual of the π^*-invariant vectors. I am grateful to Lin, as well as to Robert Sine, for correspondence on this and other matters.

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