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LOCALLY CONVEX * - ALGEBRAS

by

Steven M. MOORE

ABSTRACT

We generalize some of the known results about C*-algebras to the case of a locally convex *-algebra. In particular, we analyze the continuity of positive functionals, construct the primitive ideal space, and study a theory of representations.

RESUMEN

Generalizamos algunos de los resultados conocidos sobre las C*-álgebras al caso de una *-álgebra localmente convexa. En particular, analizamos la continuidad de los funcionales positivos, construimos el espacio de los ideales primitivos y estudiamos una teoría de representaciones.

§ *1. Introduction.* In quantum field theory in recent years a locally convex *-algebra known as the field algebra or Borchers' algebra [1] has played an increasingly prominent role. The field algebra as a topological vector space is the direct sum of *C* and the Schwartz spaces $\mathcal{S}(\mathbb{R}^{4n})$, *n=1,2,* etc. It is the author's intention to show that many of the results on *C-algebras [2] extendable to the field algebra are also extendable to the general case of certain topological *-algebras which we will call locally convex *-algebras.**

1.1. *DEFINITION.* Let $\mathfrak{A}$ be a complex *-algebra with identity 1. Suppose $\mathfrak{A}$ is a complete locally convex Hausdorff topological vector space such that the * operation is continuous and multiplication is at least separately continuous. Then $\mathfrak{A}$ is called a *locally convex *- algebra.*

When we say that multiplication is separately continuous we mean that $x \to xy$ and $x \to yx$ for fixed y are both continuous functions. This does not imply that multiplication is jointly continuous, i.e. we do not assume that $(x,y) \to xy$ is continuous. A C^*-algebra or even a Banach *-algebra is an example of a locally convex *-algebra. The field algebra is another, and for it the multiplication is not jointly continuous.

§ 2. *Positivity.* We now generalize the positivity concepts of C^*-algebras to locally convex *-algebras. First some definitions.

2.1. *DEFINITION.* $x \in \mathfrak{A}$ is *self-adjoint* if $x = x^*$. The set of all selfadjoint elements in $\mathfrak{A}$ is denoted by $\mathfrak{A}_s$. x is *positive* if $x = \sum_{n=1}^{\infty} x_n^* x_n$, where the sum is either finite or convergent in $\mathfrak{A}$. $\mathfrak{A}_+$ denotes the set of positive elements. We write $x \geq y$ if and only if $x\text{-}y$ is an element of $\mathfrak{A}_+$.

2.2. *PROPOSITION.* a. *$\mathfrak{A}_s$ is a complete real topological vector space and $\mathfrak{A}_s + i\,\mathfrak{A}_s = \mathfrak{A}$.*

b. *$\mathfrak{A}_+$ is a pointed convex cone in $\mathfrak{A}_s$.*

c. *$\mathfrak{A}_+$ is generating for $\mathfrak{A}_s$, i.e. the real subspace generated by $\mathfrak{A}_+$ is $\mathfrak{A}_s$. Since $\mathfrak{A}_+$ is a convex cone, this means that $\mathfrak{A}_s = \mathfrak{A}_+ - \mathfrak{A}_+$.*

Proof. a. The only part that merits comment is that the completeness of $\mathfrak{A}_s$ follows from the continuity of *.

b. Suppose $x \in \mathfrak{A}_+$, $x = \sum_{n=1}^{\infty} x_n^* x_n$. Then $\lambda x = \sum_{n=1}^{\infty} (\sqrt{\lambda} x_n)^* (\sqrt{\lambda} x_n)$ for $\lambda > 0$. Thus $\mathfrak{A}_+$ is a cone. Moreover $0 \in \mathfrak{A}_+$, so $\mathfrak{A}_+$ is pointed. To show that it is convex, suppose $0 < \lambda < 1$, x as above, and $y = \sum_{n=1}^{\infty} y_n^* y_n$. Then

$$\lambda x + (1-\lambda)\; y = \sum_{n=1}^{\infty} z_n^* z_n$$

where

$$z_{2k} = \sqrt{\lambda}\, x_k$$

$$z_{2k-1} = \sqrt{1-\lambda}\, y_k$$

and $k = 1, 2,$ etc.

c. Let $x \in \mathfrak{A}_s$. We have

$$x = \tfrac{1}{4}\,[\,(1+x)^*\,(1+x) - (1-x)^*\,(1-x)\,]$$

which is an element of $\mathfrak{A}_+ - \mathfrak{A}_+$. Q. E. D.

2.3. *DEFINITION.* Let ω be a continuous lineal functional on $\mathfrak{A}$, i.e. an element of $\mathfrak{A}'$. Define ω^* by

$$\omega^*(x) = \overline{\omega(x^*)}$$

Notice that ω^* is linear and that $\omega^*(x_\alpha) = \overline{\omega(x_\alpha^*)} \to 0$ if $x_\alpha \to 0$ in $\mathfrak{A}$, since ω and $*$ are continuous functions. Thus $\omega^* \in \mathfrak{A}'$ if $\omega \in \mathfrak{A}'$. ω is *self-adjoint* if $\omega = \omega^*$. The set of all self-adjoint elements of $\mathfrak{A}'$ is denoted by $\mathfrak{A}'_s$. ω is *positive* if $\omega(x) \geq 0$ for all $x \in \mathfrak{A}_+$. The set of all positive functionals in $\mathfrak{A}'$ is denoted by $\mathfrak{A}'_+$. A functional ω is a *state* if $\omega \in \mathfrak{A}'_+$ and $\omega(1) = 1$. The set of all states is denoted by E.

2.4. *PROPOSITION.* a. *$\mathfrak{A}'_s$ is a complete real topological vector space and* $\mathfrak{A}'_s + i\mathfrak{A}'_s = \mathfrak{A}'$.

b. $\mathfrak{A}'_+ \subset \mathfrak{A}'_s$.

c. *$\omega \in \mathfrak{A}'_+$ and $x, y \in \mathfrak{A}$ imply*

$$\overline{\omega(xy)} = \omega(y^*x) \tag{2.1}$$

together with the Cauchy-Schwartz inequality

$$|\omega(x^*y)|^2 \leq \omega(x^*x)\,\omega(y^*y). \tag{2.2}$$

d. *$\omega \in \mathfrak{A}'_+$ and $\omega(1) = 0$ imply $\omega = 0$.*

e. *$\mathfrak{A}'_+$ is a pointed convex cone with E as a convex base.*

Proof. a. Again this merits no comment.

b. Let $\omega \in \mathfrak{A}'_+$ · Then for $x,y \in \mathfrak{A}$, define the sesquilinear form

$$(x,y) = \omega(x^* y) \tag{2.3}$$

It is nonnegative, hence $(x,y) = \overline{(y,x)}$, which is the same as (2.1). Moreover it satisfies the Cauchy-Schwartz inequality, so $|(x,y)|^2 \leq (x,x)(y,y)$, which is (2.2).

c. Note that $x=1$ in (2.1) gives $\omega(y) = \overline{\omega(y^*)} = w^*(y)$, so $\omega \in \mathfrak{A}'_s$ if $\omega \in \mathfrak{A}'_+$.

d. For $\omega \in \mathfrak{A}'_+$ we have

$$|\omega(x)|^2 = |\omega(x \cdot 1)|^2 \leq \omega(x^* x)\ \omega(1),$$

so $\omega(1) = 0$ implies $\omega(x) = 0$ for all x.

e. Let $\omega_1, \omega_2 \in \mathfrak{A}'_+$, $0 < \lambda < 1$. Then

$$(\lambda\omega_1 + (1-\lambda)\,\omega_2)(x^* x) = \lambda\omega_1(x^* x) + (1-\lambda)\,\omega_2(x^* x) \geq 0$$

so $\mathfrak{A}'_+$ is convex. Since $0 \in \mathfrak{A}'_+$, it is pointed. Moreover, if $\omega \neq 0$ is an element of $\mathfrak{A}'_+$, then by part d, $\omega(1) \neq 0$, so that $\omega(1)^{-1}\omega$ is an element of E. Thus E is a base. It is convex because

$$\lambda\,\omega_1(1) + (1-\lambda)\,\omega_2(1) = \lambda + (1-\lambda) = 1$$

for $\omega_1, \omega_2 \in E$, and $0 < \lambda < 1$. Q.E.D.

§ 3. *Positivity and continuity.* In C^*-algebras a positive linear functional is automatically continuous. It is interesting to see what conditions we need to put on $\mathfrak{A}$ in order to insure that every positive lineal functional ω (i.e. $\omega(x) \geq 0$ for all $x \in \mathfrak{A}_+$) is automatically an element of $\mathfrak{A}'_+$ · We quote the following theorem of Schaefer [3] :

3.1. THEOREM. Let $\mathcal{E}$ be an ordered real topological vector space with positive cone C. Suppose one of the following holds :

a. *C has non-empty interior.*

b. $\mathfrak{E}$ *is metrizable and complete with* $\mathfrak{E} = C - C$.

c. $\mathfrak{E}$ *is bornological and* C *is a sequentially complete strict* $\mathfrak{B}$ *- cone. Then every positive linear functional on* $\mathfrak{E}$ *is automatically continuous.*

Conditions a. and b. are the condition applicable to C^*-algebras. In general condition b. is not applicable to locally convex *-algebras since they are generally not metrizable. Condition a. has an air of generality, but at least one important locally convex *-algebra, the field algebra, does not fulfill condition a. because $\mathfrak{A}_+$ has empty interior in that case. Thus we should try to understand condition c. more. It is easy to see that $\mathfrak{A}_s$ is an ordered real vector space with the order relation defined in 2.1. $\mathfrak{A}_s$ has positive cone $\mathfrak{A}_+$. Schaefer's theorem applies only to $\mathfrak{A}_s$ with its positive cone $\mathfrak{A}_+$, but what we really want are conditions that show that every positive lineal functional on $\mathfrak{A}$ is continuous. However, by a now standard trick, we can reduce the positivity-continuity question on $\mathfrak{A}$ to one on $\mathfrak{A}_s$. Every linear functional ω on $\mathfrak{A}$ can be decomposed as

$$\omega(x) = \omega_1(x) - i\,\omega_1(ix),$$

where ω_1 is a real linear functional on $\mathfrak{A}_s$. ω is continuous if, and only if, ω_1 is continuous. $\omega \geq 0$ if, and only if, $\omega_1 \geq 0$ on $\mathfrak{A}_s$. Thus Schaefer's theorem applies to $\mathfrak{A}$.

3.2 PROPOSITION. $\mathfrak{A}_+$ *is a strict* $\mathfrak{B}$*- cone in* $\mathfrak{A}_s$.

Proof. **We know that $\mathfrak{A}_s = \mathfrak{A}_+ - \mathfrak{A}_+$. Thus, if B is bounded in $\mathfrak{A}_s$, then**

$$B = B \cap \mathfrak{A}_s = B \cap \mathfrak{A}_+ - B \cap \mathfrak{A}_+$$

so $\mathfrak{A}_+$ is a strict $\mathfrak{B}$-cone. Q.E.D.

This proposition shows that we only have to check that $\mathfrak{A}_+$ is sequentially complete and that $\mathfrak{A}_s$ is bornological in order to apply Schaefer's condition c. to a locally convex *-algebra.

§ 4. *The primitive ideal space.* **We now want to introduce a structure space** for $\mathfrak{A}$ as the space of primitive ideals with the Jacobson topology [4]. We recall that a two-sided ideal P is primitive if it is the maximal two-sided ideal contained in some maximal left ideal I. We denote the set of primitive ideals by Prim ($\mathfrak{A}$). A closure relation defines a topology [5]. The closure relation for Prim ($\mathfrak{A}$) is defined as follows : Let $S \subset Prim(\mathfrak{A})$ and define

$$D_S = \bigcap \{ P \in Prim(\mathfrak{A}) : P \in S \}. \tag{4.1}$$

Then

$$Cl(S) = \{ P \in Prim(\mathfrak{A}) : \quad P \supset D_S \} \tag{4.2}$$

is a closure operation on $Prim(\mathfrak{A})$. $Prim(\mathfrak{A})$ is a compact space and is connected when $\mathfrak{A}$ has no non-trivial idempotents.

We note that C^*-algebras have non-trivial idempotents (the projections), so the structure space of a C^*-algebra is not connected. However the field algebra $\bigoplus_{n=0}^{\infty} \mathcal{S}(\mathbb{R}^{4n})$ has no non-trivial idempotents, so its structure space is connected. Nevertheless $Prim(\mathfrak{A})$ as a topological space in either of these cases is very complicated. Questions we would like to have the answers to are : When is $Prim(\mathfrak{A})$ T_0? T_1? T_2? How do we recover the essential structure of the algebra $\mathfrak{A}$ from $Prim(\mathfrak{A})$?

§ 5. *Representations.* The most interesting part of the theory to us is the study of the representations of a locally convex *-algebra (mainly because we have been able to obtain results in this part of the theory !). In this section we generalize the well-known GNS construction of C^*-algebras [6,7] to locally convex *-algebras. Since the algebra is not normed in general, it should not be surprisong that we have to consider unbounded operators on Hilbert space.

5.1 DEFINITION. Let $\mathcal{H}$ be a separable Hilbert space and π a map from $\mathfrak{A}$ into the closed and densely defined operators on $\mathcal{H}$ such that :

a. There is a dense set of vectors $D(\pi) \subset \mathcal{H}$ such that the operators $\pi(x)$ for $x \in \mathfrak{A}$ are all defined and closed on $D(\pi)$ and $\pi(x) D(\pi) \subset D(\pi)$.

b. If $x_\alpha \to x$ in $\mathfrak{A}$, then $(\Phi, \pi(x_\alpha)\Psi) \to (\Phi, \pi(x)\Psi)$ for all $\Phi, \Psi \in D(\pi)$

c. For all $\Phi, \Psi \in D(\pi)$, $(\Phi, \pi(x)\Psi) = (\pi(x^*)\Phi, \Psi)$, i.e. $\pi(x^*) \subset \pi(x)^*$. Thus for self-adjoint x, $\pi(x)$ is a symmetric operator (Note that we do not require that self-adjointness be preserved.)

d. Except possibly for the* operation, π preserves the algebraic operations of $\mathfrak{A}$.

Then π is called a *representation* of the locally convex *-algebra $\mathfrak{A}$.

5.2 LEMMA. Let $\omega \in E$. Define

$$L(\omega) = \{ x \in \mathfrak{A} : \omega(x^* x) = 0 \} \cdot \qquad (5.1)$$

Then $L(\omega)$ is a self-adjoint proper closed left ideal of $\mathfrak{A}$. Thus $\mathfrak{A}/L(\omega)$ is a Hausdorff space.

Proof. Let $x \in L(\omega)$, $y \in \mathfrak{A}$. Then

$$|\omega(yx)|^2 \leq \omega(yy^*)\omega(x^*x) = 0 .$$

Thus $yx \in L(\omega)$. Obviously $x^* \in L(\omega)$. Using (2.1) and taking $x, y \in L(\omega)$, we have

$$\omega((xy)^* xy) = \omega(y^* x^* xy) = \omega((y^* x^* x) y) = 0 ,$$

$$\omega((x+\lambda y)^* (x+\lambda y)) = \omega(x^* x) + \overline{\lambda}\omega(y^* x) + \lambda\omega(x^* y) + |\lambda|^2 \omega(y^* y) = 0$$

so $L(\omega)$ is a left ideal. Since $\omega \neq 0$, $1 \notin L(\omega)$, so $L(\omega)$ is proper.

Let (x_α) be a Cauchy generalized sequence in $L(\omega)$. Since $\mathfrak{A}$ is complete, $x_\alpha \to x$ for some $x \in \mathfrak{A}$. Since multiplication is separately continuous $x^* x_\alpha \to x^* x$. Thus $0 = \omega(x^* x_\alpha) \to \omega(x^* x)$, so $L(\omega)$ is closed. Q.E.D.

5.3 DEFINITION. π is a *cyclic representation* if there exists a unit vector Ω in $D(\pi)$ such that $\{\pi(x)\Omega : x \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$. In this case Ω is called a *cyclic vector.*

5.4 THEOREM. (GNS Construction) Let $\omega \in E$. Then there exists a Hilbert space $\mathcal{H}_\omega$ and a cyclic representation π_ω of $\mathfrak{A}$ with cyclic vector Ω_ω such that

$$\omega(x) = (\pi_\omega(x)\Omega_\omega, \Omega_\omega) . \qquad (5.2)$$

This representation is unique up to unitary equivalence in the sense that, if π is another cyclic representation with cyclic vector Ω such that (5.2) holds, *then there exists a unitary operator U such that* $U^*\pi(x)\, U\,\Phi = \pi_\omega(x)\,\Phi$, *for all* $x \in \mathfrak{A}$, $\Phi \in D(\pi_\omega)$.

Proof. We have just shown that $\mathfrak{A}/L(\omega)$ is a Hausdorff space. Let x_ω denote the image of x under the canonical map $\mathfrak{A} \to \mathfrak{A}/L(\omega)$. We define the following sesquilinear form on $\mathfrak{A}/L(\omega)$:

$$(x_\omega, y_\omega) = \omega\,(x^* y) \tag{5.3}$$

Now $\| x_\omega \|^2 = \omega(x^* x) = 0$ if and only if $x \in L(\omega)$, so the sesquilinear form is non-degenerate and positive. Let $h_1, h_2 \in L(\omega)$ and note that

$$\omega^*((x+h_1)^*(y+h_2)) = \omega(x^* y) + \omega(h_1^* y) + \overline{\omega(h_2^* x)} + \omega(h_1^* h_2) = \omega(x^* y),$$

since $L(\omega)$ is a left ideal. Thus the value of (x_ω, y_ω) depends only on the cosets and not their representatives. Hence $\mathfrak{A}/L(\omega)$ is a pre-Hilbert space. Let $\mathcal{H}_\omega$ be its completion, and take $\Omega_\omega = 1_\omega$. Since $1 \notin L(\omega)$, $\Omega_\omega \neq 0$, and clearly $\| \Omega_\omega \| = 1$. We take $D(\pi_\omega) = \mathfrak{A}/L(\omega)$ and define π_ω by

$$\pi_\omega(x)\, y_\omega = (xy)_\omega \tag{5.4}$$

To check that it is a representation, note that if $h \in L(\omega)$, then

$$\pi_\omega(x)\,(y_\omega + h_\omega) = \pi_\omega(x)\, y_\omega,$$

and $\pi_\omega(x)$ does not depend on the representative of the coset y_ω; that π_ω is linear and satisfies c. and d. of 5.1 is easy to check by direct calculation. Now suppose $y_\alpha \to y$ in $\mathfrak{A}$. Then

$$(x_\omega, \pi_\omega(y_\alpha)\, z_\omega) = \omega(x^* y_\alpha z) \to \omega(x^* yz) = (x_\omega, \pi_\omega(y)\, z_\omega).$$

The denseness of $\{\pi_\omega(x)\,\Omega_\omega\}$ follows from

$$\{\pi_\omega(x)\,\Omega_\omega\} = \{x_\omega : x \in \mathfrak{A}\} = \mathfrak{A}/L(\omega) = D(\pi_\omega).$$

For the uniqueness, if π satisfies (5.2), define U on $D(\pi_\omega)$ by

$$U\,\pi_\omega(x)\,\Omega_\omega = \pi(x)\,\Omega \tag{5.5}$$

Then U is linear on $D(\pi_\omega)$ and

$$\|U\pi_\omega(x)\Omega_\omega\|^2 = (\pi_\omega(x)\Omega_\omega, \pi_\omega(x)\Omega_\omega)$$

$$= (\pi_\omega(x^* x)\Omega_\omega, \Omega_\omega) = \omega(x) = (\pi(x^* x)\Omega, \Omega) = \|\pi(x)\Omega\|^2$$

shows that U extends to a unitary operator. Moreover,

$$U^* \pi(x) U \pi_\omega(x)\Omega_\omega = U^* \pi(x)\Omega = \pi_\omega(x)\Omega$$

gives the intertwining property. Q.E.D.

In what follows we will take the representation π_ω as our prototype representation, i.e., from now on, π is a cyclic representation of $\mathfrak{A}$ with cyclic vector Ω and $D(\pi) = \{\pi(x)\Omega ; x \in \mathfrak{A}\}$.

§ 6. *Irreductibility.* We take a weak form of a Schur-type criterion for the irreducibility of matrix algebras as our definition of irreducibility. This definition is borrowed from quantum field theory [8].

6.1 DEFINITION. The *commutant* of π, denoted by $\pi(\mathfrak{A})'$, contains all the bounded operators B on $\mathcal{H}$ for which

$$(\Psi, B\pi(x)\Phi) = (\pi(x^*)\Psi, B\Phi) \qquad (6.1)$$

for all $x \in \mathfrak{A}$, $\Psi, \Phi \in D(\pi)$. π is *irreducible* if $\pi(\mathfrak{A})' = \mathbb{C}\, I$.

$\pi(\mathfrak{A})'$ is a linear manifold and symmetric in $\mathcal{B}(\mathcal{H})$, the C^*-algebra of all bounded operators on $\mathcal{H}$. In general it is not an algebra, as we will see later.

6.2 DEFINITION. Let $\omega \in E$. ω is *pure* if it cannot be written as a non-trivial convex combination of states, i.e. as $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ where $0 < \lambda < 1$, $\omega_1 \neq \omega_2 \in E$.

We will see that the connection between 6.1 and 6.2 is the same as that of C^*-algebra theory [9].

6.3 LEMMA. Let b be a bilinear form on a dense domain $D \times D$, $D \subset \mathcal{H}$, such that $0 < b(\Phi, \Phi) < \|\Phi\|^2$, for $\Phi \neq 0$ in D. Then there exists a positive *operator B such that* $0 \leq B \leq I$ *and* $b(\Psi, \Phi) = (\Psi, B\Phi)$ *for* $\Phi, \Psi \in D$.

Proof. The Cauchy-Schwartz inequality gives

$$|b(\Psi,\Phi)|^2 \leq b(\Psi,\Psi)\, b(\Phi,\Phi) \leq \|\Psi\|^2 \cdot \|\Phi\|^2 .$$

Since D is dense, b extends by continuity to a bounded bilinear form on $\mathcal{H}\times\mathcal{H}$ which is symmetric. By the Riesz representation theorem, there exists a unique bounded operator B such that $b(\Psi, \Phi) = (\Psi, B\Phi)$. Since b is symmetric, B is self-adjoint. $0 \leq B \leq I$ follows from $0 \leq (\Psi, B\Psi) \leq (\Psi, \Psi)$. Q.E.D.

6.4 *THEOREM. Let* $\omega \in E$. *Then* π_ω *is irreducible if, and only of,* ω *is pure.*

Proof. Suppose ω is not pure. Then there exists $\omega_1 \neq \omega_2$, both in E, such that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $0 < \lambda < 1$. Define b by

$$b(\pi_\omega(x)\,\Omega_\omega,\ \pi_\omega(y)\,\Omega_\omega) = \lambda\omega_1(x^* y) .$$

For $x \notin L(\omega)$,

$$0 \leq b(\pi_\omega(x)\,\Omega_\omega, \pi_\omega(x)\,\Omega_\omega) = \lambda\,\omega_1(x^*x) \leq \omega(x^*x) \leq \|\pi_\omega(x)\,\Omega_\omega\|$$

so using the lemma we have $0 \leq B \leq I$, so that

$$\lambda\,\omega_1(x^* y) = (\pi_\omega(x)\,\Omega_\omega,\ B\pi_\omega(y)\,\Omega_\omega) ,$$

for all x and y. But

$$\lambda\,\omega_1(x^* y\, z) = \lambda\omega_1((y^* x)^* z) ,$$

written in terms of B, gives

$$(\pi_\omega(x)\Omega_\omega,\ B\,\pi_\omega(y)\,\pi_\omega(z)\,\Omega_\omega) = (\pi_\omega(y^*)\,\pi_\omega(x)\,\Omega_\omega,\ B\pi_\omega(z)\,\Omega_\omega) ;$$

so $B \in \pi_\omega(\mathfrak{A})'$. Now suppose $B = \mu I$. Then $\lambda\omega_1(x) = \mu\,\omega(x)$ for all x. Taking $x = I$ gives $\lambda = \mu$, or $\omega = \omega_1$. This is impossible since $\omega_1 \neq \omega_2$.

Conversely, if π_ω is not irreducible, then there exists $B \in \pi_\omega(\mathfrak{A})'$ so that $B \neq \mu I$. Without loss of generality, take $0 < B < I$ since $\pi_\omega(\mathfrak{A})'$ is a symmetric subspace of $\mathcal{B}(\mathcal{H}_\omega)$. Notice that $(\Omega_\omega, B\,\Omega_\omega) > 0$ since otherwise

$$(\pi_\omega(x)\,\Omega_\omega, B\,\pi_\omega(x)\,\Omega_\omega) = (\pi_\omega(x^*x)\,\Omega_\omega, B\,\Omega_\omega) \leq \| B^{\frac{1}{2}}\pi_\omega(x^*x)\,\Omega_\omega \| \cdot \| B^{\frac{1}{2}}\Omega_\omega \| = 0.$$

Since $D(\pi_\omega)$ is dense, this would imply $B=0$. From this observation we find

$$(\Omega_\omega, \pi_\omega(x^*x)\,B\,\Omega_\omega) = (\pi_\omega(x)\,\Omega_\omega, B\,\pi_\omega(x)\,\Omega_\omega) > 0,$$

$$(\Omega_\omega, \pi_\omega(x^*x)\,(I-B)\,\Omega_\omega) = (\pi_\omega(x)\,\Omega_\omega, (I-B)\,\pi_\omega(x)\,\Omega_\omega) > 0.$$

Define

$$\omega_1(x) = \| B^{\frac{1}{2}}\,\Omega_\omega \|^{-2} (\pi_\omega(x)\,\Omega_\omega, B\,\Omega_\omega) ;$$

$$\omega_2(x) = \| (I-B)^{\frac{1}{2}}\,\Omega_\omega \|^{-2} (\pi_\omega(x)\,\Omega_\omega, (I-B)\,\Omega_\omega).$$

Both $\omega_1, \omega_2 \in E$, and $\omega = \lambda\omega_1 + (1-\lambda)\,\omega_2$ where $\lambda = (\Omega_\omega, B\,\Omega_\omega)$. $\omega_1 = \omega_2$ would imply that $B = \mu I$, which is not so by assumption. Thus ω is not pure.

Q.E.D.

§ 7. *Self-adjoint representations.* We now concentrate on the problem of when $\pi(x^*) = \pi(x)^*$ for each x, i.e. when x self-adjoint implies $\pi(x)$ self-adjoint. This is equivalent to asking when $\pi(x^*)^* = \pi(x)$ for all x. $\pi(x^*)^* \supset \pi(x)$ always by property c. of 5.1. This suggests that we should define a new map π^* by

$$\pi^*(x)\,\Phi = \pi(x^*)^*\,\Phi \qquad (7.1)$$

and ask when $\pi = \pi^*$, and this is the final form in which we shall pose the problem. Of course, our first task is to see when π^* is a representation.

7.1 *PROPOSITION.* π^* *defined by* (7.1) *for* Φ *in*

$$D(\pi^*) = \bigcap_{x \in \mathfrak{A}} D(\pi(x)^*) \qquad (7.2)$$

satisfies all the properties of 5.1 except possibly property c. Moreover π

cyclic implies π^* *cyclic.*

Proof. We recall that $\pi(x)^*$ has domain

$$D(\pi(x)^*) = \{\Phi : |(\Phi, \pi(x)\Psi)| \leq C \cdot \|\Psi\| \text{ for some constant } C \geq 0 \text{ and all } \Psi \in D(\pi(x))\}.$$

Suppose $\Phi \in D(\pi^*)$, $\Psi \in D(\pi)$. We have

$$(\pi(x^*)\Psi, \pi^*(y)\Phi) = (\pi(x^*)\Psi, \pi(y^*)^*\Phi) = (\pi(y^*)\pi(x^*)\Psi, \Phi) = (\pi((xy)^*)\Psi, \Phi)$$
$$= (\Psi, \pi((xy)^*)^*\Phi) = (\Psi, \pi^*(xy)\Phi).$$

Now

$$|(\pi(x^*)\Psi, \pi^*(y)\Phi)| \leq C \cdot \|\Psi\|,$$

for all $\Psi \in D(\pi)$, where $C = \|\pi^*(xy)\Phi\|$, so $\pi^*(y)\Phi \in D(\pi(x^*)^*)$. This is true for all $x \in \mathfrak{A}$, so $\pi^*(y)\Phi \in D(\pi^*)$. Hence $\pi^*(y)D(\pi^*) \subset D(\pi^*)$ for all $y \in \mathfrak{A}$.

Using the above notation and results, we also have

$$(\Psi, \pi^*(xy)\Phi) = (\pi(x^*)\Psi, \pi^*(y)\Phi) = (\Psi, \pi^*(x)\pi^*(y)\Phi),$$

for all $\Psi \in D(\pi)$ and $\Phi \in D(\pi^*)$. Since $D(\pi)$ is dense, $\pi^*(xy)\Phi = \pi^*(x)\pi^*(y)\Phi$ for all $x, y \in \mathfrak{A}$, $\Phi \in D(\pi^*)$.

Obviously $D(\pi^*) \supset D(\pi)$. Thus a candidate for a cyclic vector is Ω, the cyclic vector of π. But

$$\{\pi^*(x)\Omega : x \in \mathfrak{A}\} = \{\pi(x^*)^*\Omega : x \in \mathfrak{A}\} \supset \{\pi(x)\Omega : x \in \mathfrak{A}\},$$

so $\{\pi^*(x)\Omega : x \in \mathfrak{A}\}$ is dense in $\mathcal{H}$. Q. E. D.

7.2 *DEFINITION.* A representation π of $\mathfrak{A}$ is a *self-adjoint representation* if $\pi = \pi^*$. In particular, this means $\pi(x)^* = \pi(x^*)$ for each $x \in \mathfrak{A}$.

Obviously π is self-adjoint if and only if $D(\pi) = D(\pi^*)$. What is interesting is that π^* is always self-adjoint if it satisfies property c of 5.1.

7.3 *PROPOSITION. Let* $\tilde{\pi}$ *be an extension of* π (i.e. $D(\tilde{\pi}) \supset D(\pi)$ *and* $\tilde{\pi}(x) \supset \pi(x)$ *for all* $x \in \mathfrak{A}$*). Then* $\tilde{\pi}$ *is a restriction of* π^*.

Proof. Let $x \in \mathfrak{A}$. We have $\tilde{\pi}(x^*) \supset \pi(x^*)$, so $\pi(x^*)^* \supset \tilde{\pi}(x^*)^*$. But $\tilde{\pi}$ satisfies property c, so $\tilde{\pi}(x^*)^* \supset \tilde{\pi}(x)$. Thus

$$\pi(x^*)^* \supset \tilde{\pi}(x^*)^* \supset \tilde{\pi}(x) \supset \pi(x).$$

This is true for all x, so $\tilde{\pi}$ is a restriction of π. Q. E.D.

7.4 COROLLARY. *If π^* satisfies property c. of 5.1, then π^* is a self-adjoint extension of the representation π*

Proof. $(\pi^*)^* \supset \pi^*$. But $(\pi^*)^*$ is an extension of π so $(\pi^*)^* \subset \pi^*$ by the proposition. Thus $(\pi^*)^* = \pi^*$ and π^* is self-adjoint. Q. E. D.

The next basic question related to the self-adjoint problem is : What conditions guarantee that π is self-adjoint ? Our first condition is borrowed from quantum field theory [10]. Later (see 8.4) we will give another.

7.5 *DEFINITION.* Let A be an operator on a Hilbert space $\mathcal{H}$ with domain $D(A)$ dense in $\mathcal{H}$. Let $\Phi \in D(A)$ be such that the series

$$\sum_{n=o}^{\infty} \|A^n \Phi\| \frac{z^n}{n!}$$

has a finite radius of convergence. Then Φ is called an *analytic vector* for A.

Note that implicit in this definition is the hypothesis that $A^n \Phi \in D(A)$ for each n. We now quote a theorem due to Nelson [11] ·

7.6 *THEOREM. If A is a symmetric operator with dense domain $D(A)$ and if there is a dense subset $S \subset D(A)$ of analytic vectors for A, then the closure of A is self-adjoint.*

7.7 *LEMMA. Suppose π is a cyclic representation with cyclic vector Ω. Let $x \in \mathfrak{A}$. Then Ω is an analytic vector for $\pi(x)$ if and only if there exists $c(x) > 0$ such that*

$$|(\pi(x)^n \Omega, \Omega)| \leq n!\; c(x)^n \tag{7.4}$$

Proof. (7.4) implies that the series

$$\sum_{n=o}^{\infty} \|\pi(x)^n \Omega\| \frac{z^n}{n!}$$

is majorized by

$$\sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} c(x)^n z^n$$

with the radius of convergence

$$r = (2\, c(x))^{-1} \neq 0 .$$

Conversely , if the series has finite radius of convergence, the sequence

$$\sqrt[n]{\frac{\|\pi(x)^n \Omega\|}{n!}}$$

must be bounded for $n \to \infty$. Hence there exists $c(x) > 0$ such that

$$\|\pi(x)^n \Omega\| \leq n! \cdot c(x)^n . \qquad \text{Q.E.D.}$$

7.8 THEOREM. Let π be a cyclic representation of $\mathfrak{A}$ with cyclic vector Ω with the following properties :

a. Ω is an analytic vector for each $\pi(x)$.

b. For each x, $\{\pi(y)\Omega : xy = yx\}$ is dense in $\mathcal{H}$.

Then π is a self-adjoint representation.

Proof. Let $x,y \in \mathfrak{A}$, with $x = x^*$ and $xy = yx$. We have

$$\begin{aligned}
\|\pi(x)^n \pi(y)\Omega\|^2 &= (\pi(x^{2n})\Omega, \pi(y^* y)\Omega) \\
&\leq \|\pi(x^{2n})\Omega\| \cdot \|\pi(y^*y)\Omega\| \\
&\leq (\pi(x^{4n})\Omega, \Omega)^{\frac{1}{2}} \|\pi(y^*y)\Omega\| \\
&\leq \sqrt{(4n)!} \cdot c(x)^{2n} \cdot \|\pi(y^*y)\Omega\| .
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} \|\pi(x)^n \pi(y)\Omega\| \frac{z^n}{n!}$$

is majorized by

$$\|\pi(y^*y)\Omega\|^{\frac{1}{2}} \sum_{n=0}^{\infty} \sqrt[4]{(4n)!} \cdot c(x)^n \cdot \frac{z^n}{n!} ,$$

which has the radius of convergence

$$r = (4\, c(x))^{-1} \neq 0 .$$

Thus $\pi(y)\Omega$ is an analytic vector for $\pi(x)$. $\pi(x)$ therefore has a dense set of analytic vectors. $\pi(x)$ is already closed, so it is self-adjoint. Q.E.D.

As we stated above, the condition for self-adjointness that we have given is borrowed from quantum field theory. In the work of Borchers and Zimmerman mention is made of the following example of Hamburger [12] : There is an operator A which is symmetric but not self-adjoint which has a cyclic vector Ω that satisfies $(A^n \Omega, \Omega) = O(n^{n(1+\epsilon)})$ with ϵ arbitrarily small. Thus

$$\sum_{n=0}^{\infty} (A^n \Omega, \Omega) \frac{z^n}{n!}$$

does not have a finite radius of convergence. Let $\mathfrak{A}$ be the *-algebra generated by an identity element 1 and another self-adjoint element x, 1 and x being elements of some locally convex *-algebra. Define the topology by taking the induced topology. Define π by $\pi(x^n) = A^n$ and extend by linearity. Then π is not a self-adjoint representation.

§ 8. *Properties of the commutant.* The commutant of a representation was defined in § 6. Here we want to study some of its properties. First of all, we note that Hamburger's example gives us an example where $\pi(\mathfrak{A})'$ is not an algebra.

8.1 PROPOSITION. Suppose the operator A is the one of Hamburger's example and let π and $\mathfrak{A}$ be as defined in the final paragraph of § 7. Then $\pi(\mathfrak{A})'$ is not an algebra.

Proof. Suppose $\pi(\mathfrak{A})'$ is an algebra. Let U be the partial isometry from $R(\pi(x+i))$ onto $R(\pi(x-i))$ defined by

$$U(\pi(x+i)\Phi) = \pi(x-i)\Phi .$$

Note that $U\Phi = 0$ for $\Phi \in R(\pi(x+i))^{\perp}$. (Here $R(A)$ is the range of the operator A.) We have that $U^*U = P_+$ and $UU^* = P_-$ where P_+ and P_- are thr projec -

tion onto $R(\pi(x+i))$ and $R(\pi(x-i))$ respectively. From the definition of U we have that

$$(\Phi, U\pi(x)\Psi) = (\Phi, U\pi(x+i)\Psi) - i(\Phi, U\Psi) = (\Phi, \pi(x-i)\Psi) - i(\Phi, U\Psi)$$

whereas

$$(\pi(x)\Phi, U\Psi) = (U^*\pi(x)\Phi, \Psi) = (U^*\pi(x-i)\Phi, \Psi) - i(U^*\Phi, \Psi)$$

$$= (\pi(x+i)\Phi, \Psi) - i(\Phi, U\Psi) = (\Phi, \pi(x-i)\Psi) - i(\Phi, U\Psi).$$

Hence U is an element of $\pi(\mathfrak{A})'$. Since $\pi(\mathfrak{A})'$ is symmetric, $U^* \in \pi(\mathfrak{A})'$. But $\pi(\mathfrak{A})'$ is an algebra, so $I-P_+$ an $I-P_-$ are contained in $\pi(\mathfrak{A})'$. Thus

$$(\Phi, (I-P_+)\Psi) = (\Phi, (I-P_+)\Psi) + i(\Phi, (I-P_+)\pi(x+i)\Psi)$$

$$= i(\Phi, (I-P_+)\pi(x)\Psi) = i(\pi(x)\Phi, (I-P_+)\Psi)$$

$$= i((I-P_+)\pi(x+i)\Phi, \Psi) - (\Phi, (I-P_+)\Psi)$$

$$= -(\Phi, (I-P_+)\Psi),$$

for all $\Phi, \Psi \in D(\pi)$. Since $D(\pi)$ is dense in $\mathcal{H}$, we have $P_+ = I$. A similar calculation shows $P_- = I$. Hence $\pi(x)$ has deficiency spaces

$$R(\pi(x+i))^{\perp} = R(\pi(x-i))^{\perp} = (0).$$

This means that $\pi(x)$ is self-adjoint [13]. Q. E. D.

In the proof of the proposition, we actually showed that $\pi(\mathfrak{A})'$ an algebra implies that π is self-adjoint. This has a much stroger converse : If π is self-adjoint, then $\pi(\mathfrak{A})'$ is a W^*-algebra. The proof is almost trivial : Let P be the spectral measure corresponding to $A = \pi(x)$. A bounded operator B commutes with $\pi(x)$ if and only if $BP(E) = P(E)B$ for every Borel set E in the real line. Hence $\pi(\mathfrak{A})' = \{P(E) : E \text{ Borel}\}'$ which is a W^*-algebra. We now ask if this is the case in general. That is, if π is a self-adjoint representation, is $\pi(\mathfrak{A})'$ a W^*-algebra ? The answer is positive, as we will see after a preliminary result interesting in its own right.

8.2 PROPOSITION. Suppose $B \in \pi(\mathfrak{A})'$. Then $BD(\pi) \subset D(\pi^*)$ and

$B\pi(x)\Phi = \pi^*(x) B\Phi$ *for all* $x \in \mathfrak{A}$, $\Phi \in D(\pi)$.

Proof. For all $x \in \mathfrak{A}$, $\Phi, \Psi \in D(\pi)$,

$$(\Phi, B\pi(x)\Psi) = (\pi(x^*)\Phi, B\Psi)$$

Since

$$|(\pi(x^*)\Phi, B\Psi)| \leq \|B\pi(x)\Psi\| \cdot \|\Phi\|$$

for all $\Phi \in D(\pi)$, it follows that $B\Psi \in D(\pi(x^*)^*)$ and

$$(\Phi, B\pi(x)\Psi) = (\Phi, \pi(x^*)^* B\Psi)$$

$D(\pi)$ dense implies $B\pi(x)\Psi = \pi(x^*)^* B\Psi$. Since $B\Psi \in D(\pi(x^*)^*)$ for all x, we have $B\Psi \in D(\pi^*)$. Thus $BD(\pi) \subset D(\pi^*)$ and $B\pi(x)\Phi = \pi^*(x)B\Phi$ for all $x \in \mathfrak{A}$, $\Phi \in D(\pi)$. Q. E. D.

8.3 THEOREM. Let π *be self-adjoint. Then* $\pi(\mathfrak{A})'$ *is a* W^**-algebra. Moreover for each* $B \in \pi(\mathfrak{A})'$ *we have* $BD(\pi) \subset D(\pi)$ *and* $B\pi(x)\Phi = \pi(x)B\Phi$ *for all* $x \in \mathfrak{A}$, $\Phi \in D(\pi)$.

Proof. $D(\pi) = D(\pi^*)$ and the lemma give that for $B \in \pi(\mathfrak{A})'$, $BD(\pi) \subset D(\pi)$ and $B\pi(x)\Phi = \pi(x)B\Phi$ for all $x \in \mathfrak{A}$, $\Phi \in D(\pi)$. We only need to show that $\pi(\mathfrak{A})'$ is a W^*-algebra. Since $\pi(\mathfrak{A})'$ is symmetric and weakly closed, it is sufficient to show $B_1, B_2 \in \pi(\mathfrak{A})'$ implies $B_1 B_2 \in \pi(\mathfrak{A})'$ [14]. But if $B_1, B_2 \in \pi(\mathfrak{A})'$ then

$$B_1 B_2 \pi(x)\Phi = B_2 \pi(x) B_1 \Phi = \pi(x) B_1 B_2 \Phi,$$

for all $x \in \mathfrak{A}$, $\Phi \in D(\pi)$. Q.E.D.

We conclude this section with another condition for self-adjointness.

8.4 PROPOSITION. Let $x \in \mathfrak{A}_s$ *and take* $B(x)$ *to be the commutative subalgebra of* $\mathfrak{A}$ *generated by 1 and* x. *If* $\pi(\mathfrak{B}(x))'$ *is an algebra for each* $x \in \mathfrak{A}_s$, *then* π *is a self-adjoint representation.*

Proof. Use the argument of 8.1. If $\pi(\mathfrak{B}(x))'$ is an algebra, then $\pi(x)$ is self-adjoint. Thus $\pi(x)^* = \pi(x)$ for all $x \in \mathfrak{A}_s$, so $\pi = \pi^*$. Q.E.D.

§ 9. *Covariant representations.* The action of groups on C^*-algebras has neen studied for some time and began with work by Segal [15]. The *-automorphisms of $\mathfrak{A}$, denoted $Aut_*(\mathfrak{A})$, is a group. We consider a fixed locally compact group $\mathcal{G}$ and a group morphism $\tau: \mathcal{G} \to Aut_*(\mathfrak{A})$. τ will be called the *action* of $\mathcal{G}$ on $\mathfrak{A}$. The only continuity notion we will need is the following : We say τ is a *continuous action* if $g \to \tau_g(x)$ is continuous for all $x \in \mathfrak{A}$. In what follows we will simplify notation by denoting $\tau_g(x)$ by $g(x)$.

9.1 DEFINITION. Let U be a strongly continuous representation of $\mathcal{G}$ into the unitary group of $\mathcal{B}(\mathcal{H})$ such that $U(D(\pi)) \subset D(\pi)$ and

$$U(g)\ \pi(x)\ U^*(g)\ \Phi = \pi(g(x))\ \Phi \tag{9.1}$$

for all $\Phi \in D(\pi)$. Then U is said to have the *covariant property* with respect to π and the pair (π, U) is called a *covariant representation* of the pair $(\mathfrak{A}, \mathcal{G})$.

9.2 PROPOSITION. Let π be irreducible. Then (π, U) is unique up to a multiplication by a one-dimensional representation of $\mathcal{G}$.

Proof. Consider two representations U_1 and U_2 which possess the covariant property. Then for $x \in \mathfrak{A}$, $g \in \mathcal{G}$, $\Phi \in D(\pi)$, we have

$$\pi(x)\,\Phi = \pi(g^{-1}g(x))\,\Phi = U_2(g^{-1})\pi(g(x))\,U_2^*(g^{-1})\,\Phi = U_2(g^{-1})U_1(g)\pi(x)\,U_1^*(g)U_2^*(g^{-1})\,\Phi.$$

Thus for all $\Phi \in D(\pi)$, $x \in \mathfrak{A}$, $g \in \mathcal{G}$

$$[\,U_2(g^{-1})\,U_1(g)\,]^*\ \pi(x)\,\Phi = \pi(x)\ [\,U_2^{-1}(g)\,U_1(g)\,]^*\ \Phi\,.$$

Thus $[\,U_2(g^{-1})\,U_1(g)\,]^* \in \pi(\mathfrak{A})'$. Since π is irreducible, this means

$$U_1^*(g)\ U_2(g) = \lambda(g)\, I$$

where $\lambda(g)\,I$ is a one-dimensional representation of $\mathcal{G}$. **Q. E. D.**

In the following, we again generalize some results from the theory of C^*-algebras [9].

9.3. DEFINITION. $\omega \in E$ is *invariant* if $\omega(g(x)) = \omega(x)$ for all $x \in \mathfrak{A}$, $g \in \mathcal{G}$.

9.4 THEOREM. If ω is an invariant state and τ is a continuous action, *then there is a strongly continuous representation* U_ω *of* $\mathcal{G}$ *in* $\mathcal{B}(\mathcal{H}_\omega)$ *so that* (π_ω, U_ω) *is a covariant representation. Moreover, the cyclic vector is stable :*

$$U_\omega(g)\,\Omega_\omega = \Omega_\omega .$$

Proof. To simplify the notation we drop the subscript ω. We define U on $D(\pi)$ by

$$U(g)\,\pi(x)\,\Omega = \pi(g(x))\,\Omega .$$

Then

$$U(g)\,\pi(y)\,U^*(g)\,\pi(x)\,\Omega = U(g)\,\pi(y\,g^{-1}(x))\,\Omega$$

$$= \pi(g(y\,g^{-1}(x))\,\Omega = \pi(g(y))\,\pi(x)\,\Omega ,$$

so U has the covariant property. Moreover

$$U(g)\,\Omega = U(g)\pi(1)\,\Omega = \pi(g(1))\,\Omega = \pi(1)\;\;\Omega = \Omega .$$

To see that U is unitary, we calculate

$$\| U(g)\,\pi(x)\,\Omega \|^2 = (\pi(g(x))\,\Omega, \pi(g(x))\,\Omega) = (\pi(g(x)^* g(x))\,\Omega, \Omega) =$$

$$= (\pi(g(x^*x))\,\Omega, \Omega) = \omega(g(x^*x) = \omega(x^*x) = \|\pi(x)\,\Omega\|^2 .$$

For the continuity, let $g_\alpha \to e$. Then

$$(\pi(x)\,\Omega, U(g_\alpha)\pi(y)\,\Omega) = (\pi(x)\Omega, \pi(g_\alpha(y)\,\Omega) \to (\pi(x)\,\Omega, \pi(y)\,\Omega) .$$

Thus $U(g_\alpha)$ converges weakly to I. But for unitary representations, weak convergence implies strong convergence, so we have the strong continuity. Q.E.D.

For an invariant state we thus have $U_\omega(g)\,\Omega_\omega = \Omega_\omega$. It may be that Ω_ω is not the only vector with this property. Ω_ω is unique (up to a multiplication by a complex number of modulus 1) if $\dim M_\omega = 1$ where

$$M_\omega = \{\Phi : U_\omega(g)\,\Phi = \Phi, \text{ for all } g \in \mathcal{G}\} \tag{9.2}$$

We now connect these ideas with the notion of an ergodic state.

9.5 DEFINITION. Let ω be invariant. If ω cammot be written as a non-trivial convex combination of invariant states, then ω is called an *ergodic state*.

9.6. THEOREM- *Let ω be invariant. Consider the following statements :*

a. *ω is ergodic.*

b. $\{\pi_\omega(\mathfrak{A}), U_\omega(\mathfrak{G})\}' = \mathbb{C} I$

c. *dim* $M_\omega = 1$.

Then $a \Leftrightarrow b \Leftarrow c$.

Proof. We again drop the subscript ω. We first note that for $\Phi \in D(\pi) \cap M$,

$$\omega_\Phi(x) = (\pi(x)\Omega, \Phi),$$

defines an invariant state if it is positive :

$$\omega_\Phi(g(x)) = (\pi(g(x))\Omega, \Phi) = (U(g)\pi(x)U^*(g)\Omega, \Phi) = (\pi(x)U^*(g)\Omega, U^*(g)\Phi)$$

$$= (\pi(x)\Omega, \Phi) = \omega_\Phi(x).$$

$a \Rightarrow b$. Suppose $\{\pi(\mathfrak{A}), U(g)\}' \neq \mathbb{C} I$. Construct ω_1 and ω_2 as in 6.4. Now $B\Omega$ and $(I-B)\Omega$ are elements of M since

$$(\pi(x)\Omega, U(g)A\Omega) = (\pi(x)\Omega, AU(g)\Omega) = (\pi(x)\Omega, A\Omega),$$

where A is either B or $I-B$. Thus ω_1 and ω_2 are also invariant states by our preceeding remarks.

$b. \Rightarrow a.$ Suppose ω is a non-trivial convex combination of invariant states ω_1 and ω_2. Construct B as in 6.4. Since $B \in \pi(\mathfrak{A})'$, we only need to show that $B \in U(g)'$. We calculate

$$(\pi(x)\Omega, BU(g)\pi(y)\Omega) = (\pi(x)\Omega, BU(g)\pi(y)U^*(g)\Omega)$$

$$= (\pi(x)\Omega, B\pi(g(y))\Omega) = \lambda\omega_1(x^* g(y)) = \lambda\omega_1(g(g^{-1}(x^*)y)$$

$$= \lambda\omega_2(g^{-1}(x^*)^* y) = (\pi(g^{-1}(x))\Omega, B\pi(y)\Omega)$$

$$= (U(g^{-1})\pi(x)U^*(g^{-1})\Omega, B\pi(y)\Omega) = (U(g^{-1})\pi(x)\Omega, B\pi(y)\Omega)$$

$$= (\pi(x)\Omega, U(g)B\pi(y)\Omega).$$

$c. \Rightarrow b.$ Let P be a proyection in $\{\pi(\mathfrak{A}), U(\mathfrak{G})\}'$. As above, $P\Omega$ is an element of M since

$$(\pi(x)\,\Omega, U(g)\,P\,\Omega) = (\pi(x)\,\Omega, P\,U(g)\,\Omega) = (\pi(x)\,\Omega\,,\,P\,\Omega\,).$$

Hence $P\Omega = \lambda\Omega$, **so**

$$(\pi(x)\,\Omega, P\,\pi(y)\Omega) = (\pi(y^*)\,\pi(x)\,\Omega, P\,\Omega) = (\pi(y^*)\,\pi(x)\,\Omega\,,\,P\,\Omega)$$

$$= (\pi(x)\,\Omega\,,\,\lambda\,\pi(y)\,\Omega\,),$$

for all $x, y \in \mathfrak{A}$. **Hence** $P = \lambda I$. **Q. E. D.**

§ 10. ***Concluding remarks.*** **In this paper we have only begun the study of locally convex *-algebras. There are many questions still open, some of these being indicated already in previous sections. As for covariant representations, we would like to know more about them when the state is not invariant. A lot of work has been done for the** C^***-algebra case, but nothing has been done for the general case, at least for the special case of the field algebra of quantum field theory.**

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Departamento de Física
Universidad Pedagógica Nacional / Universidad de los Andes
Bogotá D. E., Colombia, S. A.

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