EIGENVALUES OF NONSINGULAR MATRICES
AND COMBINATORIAL APPLICATIONS

by

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ABSTRACT

The purpose of this article is to present a result about eigenvalues of nonsingular matrices and to observe that this result implies a theorem of this author on combinatorial designs as well as other combinatorial results. The material presented herein lends itself well for use as an illustration of some nontrivial applications in a first course in Linear Algebra; these applications may be mentioned right after the concepts of eigenvalue and eigenvector have been defined.

Throughout the sequel, $J$ will denote the matrix having all its entries equal to $+1$, and $I$ will denote the identity matrix. Subscripts will be used whenever it is necessary or convenient to emphasize the order of a matrix; thus, $A_{m,n}$ will be an $m$ by $n$ matrix, and $A_m$ will be a square matrix of order $m$. The transpose of the matrix $A$ will be $A^T$. The scalar $\mu$ is an eigenvalue of the matrix $A^T$ with corresponding (nonzero) eigenvector $(a_1, a_2, \ldots, a_v)^T$ if

\[ A(a_1, a_2, \ldots, a_v)^T = \mu(a_1, a_2, \ldots, a_v)^T. \]
Let \( X = \{ x_1, x_2, \ldots, x_v \} \), and let \( X_s = \{ x_1, \ldots, x_v \} \) be subsets of \( X \). The subsets \( X_1, X_2, \ldots, X_v \) are said to form a \((v, k, \lambda)\)-design if each \( X_j (1 \leq j \leq v) \) has \( k \) elements; each two distinct \( X_i, X_j (1 \leq i, j \leq v) \) intersect in \( \lambda \) elements; and \( 0 \leq \lambda < k < v-1 \).

The preceding combinatorial design is completely determined by its incidence matrix; this is the \((0,1)\)-matrix \( A = [a_{ij}] \) defined by taking \( a_{ij} = 1 \) if \( x_j \in X_i \) and \( a_{ij} = 0 \) if \( x_j \notin X_i \).

Let \( 0 \leq \lambda < k < v-1 \). Then a \((0,1)\)-matrix \( A_v \) is the incidence matrix of a \((v, k, \lambda)\)-design if and only if \( AA^T = (k-\lambda)I + \lambda J \). More information about \((v, k, \lambda)\)-designs is available, for example, in [1] and [3].

**LEMMA.** Let \( A \) be a \( v \) by \( v \) nonsingular matrix, and suppose that \( k \) is an eigenvalue of \( A \) with corresponding eigenvector \((1, 1, \ldots, 1)^T\). Then \( A(a_1, a_2, \ldots, a_v)^T = \mu (1, 1, \ldots, 1)^T \) for some scalar \( \mu \) if and only if \( \mu = k^{-1} \).

**Proof.** First, it is observed that \( k \neq 0 \), for \( \det A \) is the product of the \( v \) (not necessarily distinct) eigenvalues of \( A \), and \( \det A \neq 0 \) since \( A \) is assumed nonsingular.

Now, using the hypotheses that \( k \) is an eigenvalue of \( A \) with corresponding eigenvector \((1, 1, \ldots, 1)^T\), and that \( A \) is nonsingular, one sees that \( A(a_1, a_2, \ldots, a_v)^T = (1, 1, \ldots, 1)^T \) for some scalar \( \mu \) if and only if \( A(a_1, a_2, \ldots, a_v)^T = (\mu k^{-1}, \mu k^{-1}, \ldots, \mu k^{-1})^T \).

The preceding Lemma yields the following more complete version of the Theorem in [2]:

**COROLLARY 1.** Suppose the subsets \( X_1, X_2, \ldots, X_v \) of a set \( X = \{ x_1, x_2, \ldots, x_v \} \) form a \((v, k, \lambda)\)-design. Then, except for the empty set and \( X \) itself, \( X \) contains no subset \( Y \) that intersects each \( X_j (1 \leq j \leq v) \) in the same number \( \lambda_1 \) of elements.

**Proof.** Let \( A \) be the incidence matrix of the given \((v, k, \lambda)\)-design; then \( A \)
is a \( v \) by \( v \) nonsingular matrix (for a proof of this, the reader is referred to the first 4 sentences in the proof of Theorem 2.1 on p. 103 of [3]) and \( k \) is an eigenvalue of \( A \) with corresponding eigenvector \( (1,1,\ldots,1)^T \); that is,

\[
A(1,1,\ldots,1)^T = k(1,1,\ldots,1)^T.
\]

If there exists a subset \( Y \) of \( X \) intersecting each \( X_j \) in the same number \( \lambda_1 \) of elements, then there is a vector \( (a_1,a_2,\ldots,a_v)^T \) (defined by taking \( a_j = 1 \) if \( x_j \in Y \) and \( a_j = 0 \) if \( x_j \notin Y \) for \( j=1,2,\ldots,v \)) such that

\[
A(a_1,a_2,\ldots,a_v)^T = \lambda_1(1,1,\ldots,1)^T.
\]

Now, as a consequence of the Lemma above, it follows that \( a_1 = a_2 = \ldots = a_v \); therefore each \( a_j = 0 \), or each \( a_j = 1 \); that is, \( Y \) is the empty set, or \( Y = X \).

The following three combinatorial results, all of which are mentioned in [2], are also simple consequences of the preceding Lemma (as well as of Corollary 1).

**COROLLARY 2.** (Theorem in [2]). Suppose the subsets \( X_1,X_2,\ldots,X_v \) of a set \( X = \{x_1,x_2,\ldots,x_v\} \) form a \((v,k,\lambda)\)-design. Then there does not exist another subset \( X_{v+1} \) of \( X \) such that \( X_{v+1} \) has \( k_1 \) elements and \( X_{v+1} \) intersects each \( X_j (1 \leq j \leq v) \) in \( \lambda_1 \) elements, where \( 0 < k_1 < v \) and \( 0 < \lambda_1 < k \).

**COROLLARY 3.** Suppose the subsets \( X_1,X_2,\ldots,X_v \) of a set \( X = \{x_1,x_2,\ldots,x_v\} \) form a \((v,k,\lambda)\)-design. Then there does not exist another subset \( X_{v+1} \) of \( X \) such that \( X_{v+1} \) has \( k \) elements and \( X_{v+1} \) intersects each \( X_j (1 \leq j \leq v) \) in \( \lambda \) elements.

**COROLLARY 4.** Suppose the subsets \( X_1,X_2,\ldots,X_v \) of a set \( X = \{x_1,x_2,\ldots,x_v\} \) form a \((v,k,\lambda)\)-design. Then there does not exist another subset \( X_{v+1} \) of \( X \) such that \( X_{v+1} \) has \( k_1 \) elements and \( X_{v+1} \) intersects each \( X_j (1 \leq j \leq v) \) in \( \lambda \) elements, where \( 0 < k_1 < v \).

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References


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