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APPLICATIONS OF A MAX-MIN PRINCIPLE

by

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RESUMEN

Sean H un espacio de Hilbert y X, Y dos subespacios cerrados que satisfacen: dim$X < \infty$ y H$\oplus$Y. En [6] se demostró que si f es un funcional de C^2 definido en H, tal que para cada $u \in H$, $D^2f(u)$ es acotado superiormente en X por una constante negativa y acotado inferiormente por una constante positiva en Y entonces f tiene un único punto crítico. Aquí notaremos que aún existe un punto crítico cuando la hipótesis sobre el comportamiento de $D^2f(u)$ en X se reemplaza por una condición sobre el crecimiento de f en X. Este resultado se aplica, en los teoremas 2 y 3, a la existencia de soluciones periódicas de sistemas de ecuaciones ordinarias y a un problema de Neumann no lineal.

In [6] it was shown that if f is a real C^2 function defined on a real Hilbert space H, then f has a unique critical point provided that H is the direct sum of closed subspaces X and Y with $dim\ X < \infty$ such that $D^2f(u)$ is bounded above by a negative constant on X and bounded below on Y by a positive constant for

all $u \in H$. In this note we point out that the assumption that $D^2 f(u)$ be negative definite on X may be replaced by a much weaker growth condition on the restriction of f to X to ensure existence of a critical point.

We apply this result to a class of periodic differential equations which have been studied by several writers and to a nonlinear Neumann problem.

Let H be a real Hilbert space and f a real function defined on H with a second continuous Frechet derivative. As is customary we define a continuously differentiable map $\nabla f: H \to H$ such that $f'(u)w = \langle \nabla f(u), w \rangle$ by means of the Riesz-Frechet theorem. We will denote the derivative of ∇f at $u \in H$ by $D^2 f(u)$. As is well known $D^2 f(u)$ is a self-adjoint operator defined on H (see [3, p.130-131]).

THEOREM 1: Let X and Y be two closed subspaces of H (not necessarily orthogonal) such that X is finite dimensional and $H = X \oplus Y$. *If there exists a constant* $m > 0$ *such that*

$$\langle D^2 f(u)\, y, y \rangle \geq m \, \| y \|^2 \tag{1}$$

for all $u \in H$ *and all* $y \in Y$ *and if*

$$f(x) \to -\infty \text{ as } |x| \to \infty \text{ with } x \in X \tag{2}$$

then there exists $u_o \in H$ *such that*

$$f(u_o) = \max_{x \in X} \min_{y \in Y} f(x+y) \tag{3}$$

and

$$\nabla f(u_o) = 0. \tag{4}$$

Proof.: Since the proof is only a modification of the proof of a similar theorem in [6] we only give it in part. Fix $\hat{x} \in X$ and define $g: Y \to R$ by $g(y) = f(\hat{x}+y)$. If $k \in Y$ then $\langle \nabla g(y), k \rangle = \frac{d}{dt} g(y+tk) \Big|_{t=o} = \langle \nabla f(\hat{x}+y), k \rangle$ and

$< D^2 g(y) k, k > = \frac{d^2}{dt^2} g(y+tk) \Big|_{t=o} = < D^2 f(\hat{x}+y) k, k >$. Hence by (2) for $y \in Y$, $k \in Y$

$$(5) \qquad < D^2 g(y) k, k > \geq m \|k\|^2 .$$

As is well documented (see for example [8, p.79-80]), (5) implies the existence of $\hat{y} \in Y$ such that

$$g(y) = \min_{y \in Y} g(y) \quad \text{and} \quad \nabla g(\hat{y}) = 0 .$$

Since $< \nabla g(y_1) - \nabla g(y_2), y_1 - y_2 > = < D^2 g(y_2 + s(y_1 - y_2)) (y_1 - y_2), y_1 - y_2 > \geq m \| y_1 - y_2 \|^2$ for some $s \in (0,1)$ (see [8, p.37]), ∇g can have only one zero on Y. Setting $\hat{y} = \phi(\hat{x})$ we can define a map $\phi : X \to Y$ such that

$$(6) \qquad f(x + \phi(x)) = \min_{y \in Y} f(x+y)$$

and such that

$$(7) \qquad < \nabla f(x + \phi(x)), k > = 0 ,$$

for all $k \in Y$. As in [6, p.597-598] a simple argument based on the implicit function theorem shows that $\phi(x)$ is a of class C^1. Define $G: X \to R$ by $G(x) = f(x + \phi(x))$. Setting $y = 0$ in (6) we see that $G(x) \leq f(x)$, so by (2), $G(x) \to -\infty$ as $\|x\| \to \infty$. Thus, as $\dim X < \infty$, there exists $x_o \in X$ with $G(x_o) = \max_{x \in X} G(x)$. Hence, if $h \in X$ is arbitrary, $0 = \frac{d}{dt} G(x_o + th) \Big|_{t=o} = < \nabla f(x_o + \phi(x_o)), h + \phi'(x) h > = 0$. But, as $\phi'(x_o)$ is a linear map from X to Y, $k = \phi'(x_o) h \in Y$ so by (7), $< \nabla f(x_o + \phi(x_o)), h > = 0$. If $w \in H$, $w = h + k$ with $h \in Y$ so by (7) and the above $< \nabla f(x_o + \phi(x_o)), w > = 0$. Consequently, if $u_o = x_o + \phi(x_o)$ then $\nabla f(u_o) = 0$, and from (6) and the above $f(u) = \max_{x \in X} f(x + \phi(x))) = \max_{x \in X} \min_{y \in Y} f(x+y)$. This proves the theorem.

In general there will be other critical points.

Example 1 : Periodically Perturbed Conservative Systems.

Let G be a real valued function defined on R^n and p a continuous 2π-periodic function defined on the real line with values in R^n. The existence of 2π-periodic solutions of the differential equation

$$u'' + \text{grad}\, G(u) = p(t) = p(t+2\pi) \tag{8}$$

has been investigated by several writers (see, for example [1], [4], [5], [7] and [9]). In [1] and [5] it is assumed that if

$$M(x) = \left(\frac{\partial^2 G}{\partial x_i \partial x_j}\right)(x) \tag{9}$$

then there exist symmetric matrices A and B, with $A \leq M(x) \leq B$ for all x, such that if $\lambda_k(A)$ ($\lambda_k(B)$) denotes the k-th eigenvalue of $A(B)$ then $[\lambda_k(A), \lambda_k(B)]$ does not contain the square of a non-negative integer for $k=1, \ldots, n$. Under this condition there exists a unique 2π-periodic solution of (8).

As an application of Theorem 1 we derive a condition which only requires a one sided bound on the Hessian matrix and a growth condition on G. Let H denote the real Hilbert space of 2π-periodic R^n valued functions with components which are absolutely continuous and have square integrable derivatives on bounded intervals with inner product

$$\langle u, v \rangle = \int_0^{2\pi} [(u'(t), v'(t)) + (u(t), v(t))]\, dt. \tag{10}$$

Here (,) denotes the usual R^n inner product. If we denote the set of constant functions by X we can write $H = X \oplus Y$ where $y(t) \in Y$ iff

$$\int_0^{2\pi} y(t)\, dt = 0.$$

If $y \in Y$ and

$$y(t) = \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt, \qquad a_k, b_k \in R^n$$

then

$$(11) \qquad \int_0^{2\pi} (y(t), y(t))\, dt = \pi \sum_{k=1}^{\infty} (\|a_k\|^2 + \|b_k\|^2) \leq$$

$$\pi \sum_{k=1} k^2 (\|a_k\|^2 + \|b_k\|^2) = \int_0^{2\pi} (y'(t), y'(t))\, dt$$

THEOREM 2. Suppose (i) : *there exists a number* $\gamma < 1$ *such that* $\langle a, M(x)a \rangle \leq \gamma \|a\|^2$ *for all* $a \in R^n$ *and all* $x \in R^n$: (ii) : $G(x) \to +\infty$ *as* $\|x\| \to \infty$; *and* (iii) $\int_0^{2\pi} p(t)\, dt = 0.$

Assertion : There exists at least one 2π *-periodic solution of* (8).

Proof : **If we define** $f: H \to R$ **by**

$$f(u) = J[u] = \int_0^{2\pi} \left[\frac{(u'(t), u'(t))}{2} - G(u(t)) + (p(t), u(t)) \right] dt$$

then by using the well-known fact that the imbedding of H **into** $C([0, 2\pi], R^n)$ **is continuous (see for example** [2, p. 26]) **we establish that** $f \in C^2$ **and**

$$(12) \qquad \langle \nabla f(u), w \rangle = \frac{d}{dt} f(u + tw) \Big|_{t=0} =$$

$$\int_0^{2\pi} [(u'(t), w'(t)) - (\text{grad } G(u(t)), w(t)) + (w(t), p(t))]\, dt,$$

$$(13) \qquad \langle D^2 f(u) v, w \rangle = \frac{d}{dt} \langle \nabla f(u + tw), v \rangle \Big|_{t=0} =$$

$$\int_0^{2\pi} [(v'(t), w'(t)) - (M(u(t))\, v(t), w(t))]\, dt,$$

for all $v \in H$ **and** $w \in H$. **From (12) we see that if** u_o **is a solution of (8) then**

$\nabla f(u_o) = 0$ and conversely if $\nabla f(u_o) = 0$ it follows from (12) and a classical integration by parts argument that $u_o \in C^2$ satisfies (8). If $u \in H$ and $k \in Y$ then by condition (i) and (13)

$$< D^2 f(u) k, k > \geq \int^{2\pi} (|| k'(t)||^2 - \gamma || k(t) ||^2) dt .$$

Hence, according to (11) and (10)

$$< D^2 f(u) k, k > \geq (1-\gamma) \int_0^{2\pi} || k'(t) ||^2 dt \geq$$

$$(\frac{1-\gamma}{2}) \int_0^{2\pi} (||k'(t)||^2 + || k(t) ||^2) dt = m < k, k >$$

with $m > 0$ so (1) holds. Consider a constant function $x \in X$. It follows from (iii) that $f(x) = \int_0^{2\pi} (-G(x) + (p(t), x)) dt = -2\pi G(x)$; so from (ii) $f(x) \to -\infty$ as $|| x || \to \infty$ and condition (2) is established. Thus there exists u_o such thal $\nabla f(u_o) = 0$ and by a previous remark u_o is a solution of (8).

Example 2 : A Neumann Problem.

We discuss very briefly the problem of existence of weak solutions of the boundary value problem

$$(14) \qquad \Delta u + g(u) = p(x) \quad \text{in } U$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial U .$$

Here U is a bounded domain in R^n with sufficiently regular boundary, Δ is the Laplacian, g is a continuously differentiable function which will satisfy other conditions, and $p \in L^2(U)$.

As our Hilbert space H we take $H = H^1(U)$ which is the completion of the inner product space $C^1(\overline{U}, R)$ with inner product

$$< u,v > = \int [(\nabla u, \nabla v) + uv] \, dx .$$

By a weak solution u_o of (14) we understand a element of $u_o \in H$ such that for all $w \in H$

$$\int [(\nabla u_o, \nabla w) - g(u_o) w + p(x) w] \, dx = 0.$$

With enough regularity assumptions on g, p, and ∂U, u_o will satisfy the diffe - rential equation and the natural boundary condition. If X denotes the set of constant functions on U we can write $H = X \oplus Y$ where

$$y \in Y \quad \text{iff} \int y \, dx = 0 .$$

If $0 = \lambda_o < \lambda_1 \leq \lambda_2 \leq \cdots$ denote the eigenvalues of the linear problem $\nabla u + \lambda u = 0$ in U $\frac{\partial u}{\partial n} = 0$ on ∂U, then by the well known characterization of λ_1.

$$\int y^2(x) \, dx \leq \left(\frac{1}{\lambda_1}\right) \int (\nabla y(x), \nabla y(x)) \, dx \tag{15}$$

for all $y \in Y$. Let us assume that

$$- \infty < M \leq g' \leq \gamma < \lambda_1 . \tag{16}$$

If f is the functional defined on H by

$$f(u) = \int \left(\frac{(\nabla u, \nabla u)}{2} - G(u) + p(x) \, u \right) dx$$

where $G(s) = \int_o^s g(x) \, dx$, then

$$< \nabla f(u), w > = \int [(\nabla u, \nabla w) - g(u) w + p(x) w] \, dx$$

so solutions of $\nabla f = 0$ coincide with weak solutions of (14). By imitating the argument given in the previous section, using (15) and (16), we see that if $u \in H$, $k \in Y$

$$\langle D^2 f(u)\, k,k \rangle = \int [\,(\nabla k, \nabla k) - g(u)\, k^2\,]\, dx \geq$$

$$m\,||\,k\,||^2 \;,\; m = \frac{1}{2}\,(1 - \frac{\gamma}{\lambda_1}) > 0 \;.$$

From this we easily conclude

THEOREM 3. If $p \in Y$, condition (16) holds, and $G(x) \to \infty$ as $|x| \to \infty$ then there exists a weak solution of 14.

Actually the condition that g' be bounded below can be omitted if g' satisfies a suitable polynomial growth condition.

References

1. S. Ahmad, *An existence theorem for periodically perturbed conservative systems*, Michigan Math. J. 20(1973), 385-392.
2. G. Fichera, *Linear Elliptic Differential Systems and Eigenvalue Problems*, Lecture Notes in Mathematics 8, Springer, Berlin, Heidelberg, New York, 1965.
3. A. Goldstein, *Constructive Real Analysis*, Harper and Row, New York, Evanston, and London, 1967.
4. R. Kannan, *Existence of periodic solutions of nonlinear differential equations*, Trans. Amer. Math. Soc. 217 (1976) 225-236.
5. A. Lazer, *Application of a lemma on bilinear forms to a problem in nonlinear oscillations*, Proc. Amer. Math. Soc. 33(1972), 89-94.
6. A. Lazer, E. Landesman, and D. Meyers, *On saddle point problems in the calculus of variations, the Ritz algorithm, and monotone converfence*, J. Math. Anal. Appl. 53(1975), 594-614.
7. A. Lazer and D. Sanchez, *On periodically perturbed conservative systems*, Mich. Math. J. 16 (1969), 193-200.

8. *M. Vainberg, Variation Methods for the Study of Nonlinear Operators,* Holden-Day, San Francisco, 1964.

9. *S. N. Chow, J. K. Hale, J. Mallet Paret, Applications of Generic Bifurcation I,* Arch. Rat. Mech. Anal., to appear.

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