QUASI-COVARIANT REPRESENTATIONS

OF NUCLEAR $\ast$-ALGEBRAS

por

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ABSTRACT

We consider the extension of the concept of a quasi-covariant representation of C*-algebras to nuclear $\ast$-algebras. Necessary conditions for a representation to be quasi-covariant are obtained.

RESUMEN

Consideramos la extensión del concepto de una...
representación cuasi-covariante de $C^*$-álgebras a
*álgebras nucleares. Condiciones necesarias para
que una representación sea cuasi-covariante son ob-
tenidas.

§ Introducción.

In [1] we introduced locally convex *-algebras. Although this is a new type of topological algebraic
structure that is ripe for more work, it has become
clear that stronger properties are needed in order
to get substantial results. Since a $C^*$-algebra is
a nuclear *-algebra if and only if it is finite di-
mensional [2], one might expect that the additional
hypothesis of nuclearity would be interesting. Thus
here we consider nuclear *-algebras, i.e. a locally
convex *-algebra $\mathcal{A}$ that is also a nuclear space.
This still includes the physically interesting case
of the field algebra [3].

For nuclear *-algebras it is not possible to
define quasi-equivalent representations in the same
way as in the $C^*$-algebra theory [4] (e.g. in the
field algebra [3], all projections are trivial).
But Kadison [5] has given an equivalent definition
using the following concepts: Let $\Pi$ be a represen-
tation of $\mathcal{A}$ in the sense of [1]. $\omega_\Phi$ is a vector
state of $\Pi$ if $\omega_\Phi(\lambda) = (\Phi, \Pi(\lambda)\Phi)$ where $\Phi \in \mathcal{D}(\Pi)$,
$\|\Phi\| = 1$. The set of all vector-states of $\Pi$ is deno-
ted by $E(\Pi)$ and the closure of the convex hull of
E(Π) by F(Π) (closure in the weak topology). A representation Π₁ is quasi-equivalent to a representation Π₂ if \( F(\Pi_1) = F(\Pi_2) \).

§ 2. Quasi-covariant representations.

In [1] we also introduced the concept of covariant representation. We say that a representation Π is quasi-covariant if it is quasi-equivalent to \( \Pi^- \), where \((\Pi^-, V^-)\) is some covariant representation of \((\mathcal{A}, \Omega)\).

We remember that our working hypothesis is that \( g \rightarrow gx \) is continuous for each \( x \in \mathcal{A} \). The question of the continuity of \( g \rightarrow g\omega \) is more delicate, partly because of possible ambiguities in the topology of \( \mathcal{A}^- \). There is a large class of topologies for \( \mathcal{A}^- \) for which \((\mathcal{A}, \mathcal{A}^-)\) is a dual pair. Among these are the weak topology and the strong topology [2]. In analogy with the C*-algebra case [6,7], one might be tempted to elect the strong topology. However, for the field algebra [3], the fact that the \( \omega \) are products of tempered distributions and that we are in general treating a nuclear *-algebra which possesses very different properties than those of a C*-algebra suggests that we should consider instead the weak topology. Thus we let \( E^c \) be the set of all states such that \( g \rightarrow g\omega \) is continuous with respect to the weak topology on \( \mathcal{A}^- \).
2.1 Theorem. Let \((\mathcal{A}, \mathcal{G})\) be a covariant representation of \((\mathcal{A}, \mathcal{G})\). Then \(E(\mathcal{A}) \subset E^c\)

Proof. Let \(\phi \in F(\mathcal{A}), \|\phi\| = 1\). Then \(g\omega_\phi (x) = (\phi, \Pi(g \cdot x) \cdot \phi)\). Thus \(|g\omega_\phi (x) - \omega_\phi (x)| = |(\phi, \Pi(g \cdot x - x) \cdot \phi)|\). Since \(g \to g \cdot x\) is continuous, \(g \to x\) when \(g \to e\). Thus \((\phi, \Pi(g \cdot x - x) \cdot \phi) \to 0\) when \(g \to e\). QED.

2.2 Theorem. \(E^c\) has the following properties:

a. \(E^c\) is convex.

b. \(E^c\) is weakly closed.

c. \(E^c\) is invariant with respect to \(\mathcal{G}\), i.e.
\[ gE^c = E^c \] for all \(g \in \mathcal{G}\).

Proof. a. Let \(\omega_1, \omega_2 \in E^c\) and \(0 \leq \lambda \leq 1\), Then
\[
|g(\lambda \omega_1 + (1-\lambda)\omega_2)(x) - (\lambda \omega_1 + (1-\lambda)\omega_2)(x)|
\]
\[
= |\lambda (g\omega_1 - \omega_1)(x) + (1-\lambda)(g\omega_2 - \omega_2)(x)|
\]
\[
\leq \lambda |(g\omega_1 - \omega_1)(x)| + (1-\lambda)|g\omega_2 - \omega_2)(x)|.
\]

b. Suppose \(\omega_\beta \to \omega\) and \(g_\alpha \to e\). We have
\[
|(g_\alpha \omega - \omega)(x)| \leq |g_\alpha (\omega - \omega_\beta)(x)| + |(g_\alpha \omega_\beta - \omega_\beta)(x)| + |(\omega_\beta - \omega)(x)|.
\]

Fix \(x\) for the moment. Since \(g \to g\omega(x)\) is continuous, we can find \(\beta_0\) such that \(\beta \geq \beta_0\) implies
\[
|\omega_\beta - \omega)(x)| \leq \varepsilon/6.
\]
Now \(\psi: g \to g (\omega - \omega_\beta)(x) = (\omega - \omega_\beta)(g \cdot x)\) is a conti-
Consider

$$I(\beta) = \left[ c(\beta) - \epsilon/6 \right] + \left[ c(\beta) + \epsilon/6 \right]$$

where $c(\beta) = (\omega - \omega_\beta)(x) = \psi(e)$. $\psi^{-1}I(\beta) = V(\beta)$ is then a neighborhood of $e$ in $\mathcal{G}$. There exists $\alpha(\beta)$ such that $\alpha > \alpha(\beta)$ implies $g_\alpha \in V(\beta)$ since $g_\alpha \to e$. If $\beta > \beta_0$, then $|c(\beta)| < \epsilon/6$, so for $\alpha > \alpha(\beta_0)$, $|\psi(g_\alpha) - \psi(e)| < \epsilon/6$, i.e.

$$|g_\alpha(\omega - \omega_\beta)(x)| < \epsilon/6 + |(\omega - \omega_\beta)g(x)| < \epsilon/3.$$ 

Fix $\beta > \beta_0$. There exists $\alpha_1$ such that $\alpha > \alpha_1$ implies $|(g_\alpha \omega_\beta - \omega_\beta)(x)| < \epsilon/3$. Thus for $\alpha > \alpha_1$ and $\alpha > \alpha(\beta)$ we have $|(g_\alpha \omega - \omega)(x)| < \epsilon$.

c. To show $h \omega \in E^C$, if $\omega \in E^C$, let $g_\alpha \to e$. Then $h^{-1}g_\alpha h \to e$. Hence $h^{-1}g_\alpha h\omega \to \omega$. $h \to h\omega(x)$ continuous implies $h(h^{-1}g_\alpha h)(\omega(x)) = g_\alpha h\omega(x) \to h\omega(x)$. 

QED.

§ 3. Necessary conditions for a quasi-covariant representation.

We have obtained the following necessary conditions for a quasi-covariant representation:

3.1 Theorem. Let $\Pi$ be a quasi-covariant representation. Then the following conditions are satisfied:

a. $F(\Pi)$ is invariant.

b. $F(\Pi) \subseteq E^C$. 

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Proof: Let $\Pi$ be a quasi-covariant representation. Then there exists a covariant representation $(\Pi_1, V_1)$ of $(\mathcal{A}, \mathcal{J})$ to which $\Pi$ is quasi-equivalent. For $\phi \in \mathcal{D}(\Pi_1), ||\phi|| = 1$,

$$g\omega_\phi(x) = \omega_\phi(gx) = (\phi, \Pi_1(gx)\phi)$$

$$= (V^* (g) \phi, \Pi_1(x) V^* (g) \phi)$$

$$= \omega V^* (g) \phi (x)$$

This means that $E(\Pi_1)$ is invariant. Thus the convex hull of $E(\Pi_1)$ is invariant. Since $E(\Pi_1) \subseteq E^C$, it follows that the closure of the convex hull is invariant. Thus $gF(\Pi_1) = F(\Pi_1)$. But $F(\Pi) = F(\Pi_1)$, so part a follows.

Now let $\omega \in E(\Pi_1)$. Then there exists $\phi \in \mathcal{D}(\Pi_1), ||\phi|| = 1$ with $\omega = \omega_\phi$. Hence

$$|g\omega(x) - \omega(x)| = |\omega(gx - x)| = |(\phi, \Pi_1(gx - x) \phi)|$$

Since $gx \to x$ is continuous, $gx \to x$ when $g \to e$. Thus $(\phi, \Pi_1(gx - x) \phi) \to 0$ when $g \to e$. Hence $\omega \in E^C$. Thus $E(\Pi_1) \subseteq E^C$. $E^C$ convex and closed implies that $F(\Pi) = F(\Pi_1) \subseteq E^C$. QED.

It is not known whether these conditions are also sufficient as they are in the $C^*$-algebra case [7].

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