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QUASI-COVARIANT REPRESENTATIONS OF NUCLEAR $*$-ALGEBRAS

por

Steven M. MOORE

ABSTRACT

We consider the extension of the concept of a quasi-covariant representation of C*-algebras to nuclear *-algebras. Necessary conditions for a representation to be quasi-covariant are obtained.

RESUMEN

Consideramos la extensión del concepto de una

representación cuasi-covariante de C*-álgebras a *-álgebras nucleares. Condiciones necesarias para que una representación sea cuasi-covariante son obtenidas.

§ Introducción.

In [1] we introduced locally convex *-algebras. Although this is a new type of topological algebraic structure that is ripe for more mork, it has become clear that stronger properties are needed in order to get substantial results. Since a C*-algebra is a nuclear *-algebra if and only if it is finite dimensional [2] , one might expect that the additional hypothesis of nuclearity would be interesting. Thus here we consider nuclear *-algebras, i.e. a locally convex *-algebra $\mathfrak{A}$ that is also a nuclear space. This still includes the physically interesting case of the field algebra [3] .

For nuclear *-algebras it is not possible to define quasi-equivalent representations in the same way as in the C*-algebra theory [4] (e.g. in the field algebra [3] , all projections are trivial). But Kadison [5] has given an equivalent definition using the following concepts: Let Π be a representation of $\mathfrak{A}$ in the sense of [1] . ω_Φ is a <u>vector state</u> of Π if $\omega_\Phi(x) = (\Phi, \Pi(x)\Phi)$ where $\Phi \in \mathcal{D}(\Pi)$, $\|\Phi\| = 1$. The set of all vector-states of Π is denoted by $E(\Pi)$ and the closure of the convex hull of

$E(\Pi)$ by $F(\Pi)$ (closure in the weak topology). A representation Π_1 is quasi-equivalent to a representation Π_2 if $F(\Pi_1) = F(\Pi_2)$.

§ 2. Quasi-covariant representations.

In [1] we also introduced the concept of covariant representation. We say that a representation Π is quasi-covariant if it is quasi-equivalent to Π', where (Π', V') is some covariant representation of $(\mathfrak{A}, \mathcal{G})$.

We remember that our working hypothesis is that $g \to gx$ is continuous for each $x \in \mathfrak{A}$. The question of the continuity of $g \to g\omega$ is more delicate, partly because of possible ambiguities in the topology of $\mathfrak{A}'$. There is a large class of topologies for $\mathfrak{A}'$ for which $(\mathfrak{A}, \mathfrak{A}')$ is a dual pair. Among these are the weak topology and the strong topology [2] . In analogy with the C*-algebra case [6,7], one might be tempted to elect the strong topology. However, for the field algebra [3] , the fact that the ω are products of tempered distributions and that we are in general treating a nuclear *-algebra which possesses very different properties than those of a C*-algebra suggests that we should consider instead the weak topology. Thus we let E^c be the set of all states such that $g \to g\omega$ is continuous with respect to the weak topology on $\mathfrak{A}'$.

2.1 Theorem. Let (Π, V) be a covariant representation of $(\mathfrak{A}, \mathcal{G})$. Then $E(\Pi) \subset E^c$

Proof. Let $\Phi \in \mathfrak{D}(\Pi)$, $\|\Phi\| = 1$. Then $g\omega_\Phi(x) = (\Phi, \Pi(g x)\Phi)$. Thus $|g\omega_\Phi(x) - \omega_\Phi(x)| = |(\Phi, \Pi(gx-x)\Phi)|$. Since $g \to gx$ is continuous, $gx \to x$ when $g \to e$. Thus $(\Phi, \Pi(gx - x)\Phi) \to 0$ when $g \to e$. QED.

2.2 Theorem. E^c has the following properties:

a. E^c is convex.

b. E^c is weakly closed.

c. E^c is invariant with respect to $\mathcal{G}$, i.e. $gE^c = E^c$ for all $g \in \mathcal{G}$.

Proof. a. Let $\omega_1, \omega_2 \in E^c$ and $0 \leqslant \lambda \leqslant 1$, Then

$$|g(\lambda\omega_1 + (1-\lambda)\omega_2)(x) - (\lambda\omega_1 + (1-\lambda)\omega_2)(x)|$$
$$= |\lambda(g\omega_1 - \omega_1)(x) + (1-\lambda)(g\omega_2 - \omega_2)(x)|$$
$$\leqslant \lambda |(g\omega_1 - \omega_1)(x)| + (1-\lambda)|(g\omega_2 - \omega_2)(x)|.$$

b. Suppose $\omega_\beta \xrightarrow[w]{} \omega$ and $g_\alpha \to e$. We have

$$|(g_\alpha\omega - \omega)(x)| \leqslant |g_\alpha(\omega - \omega_\beta)(x)| + |(g_\alpha\omega_\beta - \omega_\beta)(x)| + |(\omega_\beta - \omega)(x)|.$$

Fix x for the moment. Since $g \to g\omega(x)$ is continuous, we can find β_o such that $\beta \geqslant \beta_o$ implies $|(\omega_\beta - \omega)(x)| \leqslant \varepsilon/6$.

Now $\psi: g \to g(\omega - \omega_\beta)(x) = (\omega - \omega_\beta)(g x)$ is a continous function. Consider

$$I(\beta) =]c(\beta) - \varepsilon/6 \ , \ c(\beta) + \varepsilon/6[$$

where $c(\beta) = (\omega-\omega_\beta)(x) = \psi(e)$. $\psi^{-1} I(\beta) = V(\beta)$ is then a neighborhood of e in $\mathcal{G}$. There exists $\alpha(\beta)$ such that $\alpha \geqslant \alpha(\beta)$ implies $g_\alpha \in V(\beta)$ since $g_\alpha \to e$. If $\beta \geqslant \beta_o$, then $|c(\beta)| \leqslant \varepsilon/6$, so for $\alpha \geqslant \alpha(\beta_o)$, $|\psi(g_\alpha) - \psi(e)| \leqslant \varepsilon/6$, i.e.

$$|g_\alpha(\omega-\omega_\beta)(x)| \leqslant \varepsilon/6 + |(\omega-\omega_\beta)g(x)| \leqslant \varepsilon/3.$$

Fix $\beta \geqslant \beta_o$. There eixists α_1 such that $\alpha \geqslant \alpha_1$ implies $|(g_\alpha \omega_\beta - \omega_\beta)(x)| \leqslant \varepsilon/3$. Thus for $\alpha \geqslant \alpha_1$ and $\alpha \geqslant \alpha(\beta)$ we have $|(g_\alpha \omega - \omega)(x)| \leqslant \varepsilon$.

c. To show $h\omega \in E^c$, if $\omega \in E^c$, let $g_\alpha \to e$. Then $h^{-1} g_\alpha h \to e$. Hence $h^{-1} g_\alpha h\omega \to \omega$. $h \to h\omega(x)$ continuous implies $h(h^{-1} g_\alpha h)\omega(x) = g_\alpha h\omega(x) \to h\omega(x)$.

QED.

§ 3. <u>Necessary conditions for a quasi-covariant representation</u>.

We have obtained the following necessary conditions for a quasi-covariant representation:

3.1 <u>Theorem</u>. Let Π be a quasi-covariant representation. Then the following conditions are satisfied:

a. $F(\Pi)$ is invariant.

b. $F(\Pi) \subset E^c$.

Proof: Let Π be a quasi-covariant representation. Then there exists a covariant representation (Π_1, V_1) of $(\mathcal{A}, \mathcal{G})$ to which Π is quasi-equivalent. For $\Phi \in \mathcal{D}(\Pi_1), \|\Phi\| = 1$,

$$g\omega_\Phi(x) = \omega_\Phi(gx) = (\Phi, \Pi_1(gx)\Phi)$$
$$= (V^*(g)\Phi, \Pi_1(x) V^*(g)\Phi)$$
$$= \omega_{V^*(g)\Phi}(x)$$

This means that $E(\Pi_1)$ is invariant. Thus the convex hull of $E(\Pi_1)$ is invariant. Since $E(\Pi_1) \subset E^c$, it follows that the closure of the convex hull is invariant. Thus $gF(\Pi_1) = F(\Pi_1)$. But $F(\Pi) = F(\Pi_1)$, so part a follows.

Now let $\omega \in E(\Pi_1)$. Then there exists $\Phi \in \mathcal{D}(\Pi_1)$, $\|\Phi\| = 1$ with $\omega = \omega_\Phi$. Hence

$$|g\omega(x) - \omega(x)| = |\omega(gx - x)| = |(\Phi, \Pi_1(gx-x)\Phi)|$$

Since $gx \to x$ is continuos, $gx \to x$ when $g \to e$. Thus $(\Phi, \Pi_1(gx - x)\Phi) \to 0$ when $g \to e$. Hence $\omega \in E^c$. Thus $E(\Pi_1) \subseteq E^c$. E^c convex and closed implies that $F(\Pi) = F(\Pi_1) \subset E^c$. QED.

It is not known whether these conditions are also sufficient as they are in the C*-algebra case [7].

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Departamento de Física
Universidad de los Andes
Apartado Aéreo 4976, Bogotá 1, DE, Colombia, S.A.

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