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# ASYMPTOTIC FORM FOR GENERALIZED FACTORIAL

by

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ABSTRACT

In this note we generalize the concept of factorial by defining

$$f(n)? = \prod_{i=1}^{n} f(i)$$

for suitable f(x)'s. We then obtain an asymptotic expression, as follows

$$f(n)? \approx \sqrt{f(n)}\, e^{\sigma(n)}\, e^{\gamma}$$

with

$$\sigma(x) = \int_1^x \ln f(t)dt, \text{ and } -\sigma(\tfrac{3}{2}) \leq \gamma \leq 0.$$


RIASSUNTO

In questa nota generaliziamo il concetto del fattoriale definendo

$$F(n)? = \prod_{i=1}^{n} f(i)$$

per funzioni appropriate. Otteniamo quindi una expressione asintotica, come segue

$$f(n)? \approx \sqrt{f(n)}\, e^{\sigma(n)}\, e^{\gamma}$$

con

$$\sigma(x) = \int_1^x \ln f(t)\, dt, \text{ e } -\sigma(\tfrac{3}{2}) \leq \gamma \leq 0.$$


## § 1 Introduction.

In this note we generalize the concept of the factorial function in a novel way. An asymptotic expression along the lines of Stirling's formula, is obtained; such generalization was required to solve in close form a number of combinatorial problems the author has encountered in his work.

**Definition 1**: A continuous, monotonically increasing function f(x), from the reals R into the reals, is called a factorial generator.

Definition 2: By the generalized factorial on f, where f(x) is a factorial generator, we mean a functional

$$\mu_f : I \to R$$

with

$$\mu_f(n) = \prod_{i=1}^{n} f(i),$$

where I are the natural numbers.

We shall use the notation $f(n)? = \mu_f(n)$. Clearly, with f(x) = x, one obtains the standard factorial function. Generalized factorials with simple f(x) have frequent applications as combinatorial quantities, e.g., f(x) = 2x gives f(n)? = 2(n)!! ; f(x) = c gives $f(n)? = c^n$; etc. Also, they have interesting applications in analysis. For example,

a. [Spiegel, 63] allows us to say that if |x|<1, then
$$(1-x)^{1/2} = 1 - \sum_{i=1}^{\infty} \frac{x^i}{2i} f(i)? \quad \text{with} \quad f(x) = \frac{2x-1}{2x}$$

b. Wallis formula (see [Spivak, 67] can be written as
$$\frac{\Pi}{2} = \lim_{n\to\infty} f(n)? \quad \text{with} \quad f(x) = \frac{x^2}{x^2-1/4}$$

c. Using [CRC, 66] we can write, for example
$$\int_0^{\pi/2} \sin^{2n+1} x \, dx = h(n)? \text{ with } h(x) = \frac{2x}{2x+1};$$

$$\int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\Pi}{2} g(n)? \text{ with } g(x) = \frac{2x-1}{2x};$$

$$\int_0^{1} (1-x^2)^n = f(n)? \text{ with } f(x) \text{ given above};$$

$$\int_o^1 \frac{1}{(1+x^2)^2}\, dx = \frac{\Pi}{2}\, g\ (n-1)?$$ with $g(x)$ given above.

Other applications are readily available.

The following basic properties are easily established.

<u>Proposition 1:</u>

(i) If $f(x) = g(x)\ h(x)$ then $f(x)? = g(n)?\ h(n)?$;

(ii) If $f(x) = g(x)/h(x)$ then $f(n)? = g(n)?/h(n)?$;

(iii) If $f(x)=c^{g(x)}$ then $f(n)? = c^{\sum_1^n g(m)}$;

(iv) If $f(x) = (g(x))^c$ then $f(n)? = (g(n)?)^c$;

(v) If $f(x) = h(x) + k(x)$ then $f(n)? = \sum \prod_{i=1}^{n} b(i)$

where $b(x) = h(x)$ or $b(x) = k(x)$, and the sum is taken over all possible $2^n$ combinations.

For simple functions, the generalized factorial can be expressed in terms of the standard factorial; for example,

<u>Proposition 2:</u> Let $f(x) = ax^p$. Then $f(n)? = a^n(n!)^p$.

<u>Proof</u>: We have

$$\prod_{i=1}^{n} f(i) = \prod_{i=1}^{n} ai^p = a^n \prod_{i=1}^{n} i^p = a^n\Big(\prod_{i=1}^{n} i\Big)^p = a^n(n!)^p.$$

QED.

**Proposition 3**: Let $f(x)=a_p x^p + a_{p-1} x^{p-1} + \dots + a_1 x + a_0$. then

$$f(n)? = \sum_{0 \leq k_0, k_1, \dots, k_j, \dots, k_{n-1} \leq p} \prod_{j=0}^{n-1} a_{p-k_j} (n-j)^{p-k_j}$$

**Proof**: By definition,

$$f(n)? = (a_p n^p + a_{p-1} n^{p-1} + \dots)(a_p (n-1)^p + a_{p-1} (n-1)^{p-1} + \dots)\dots$$

Tedious collection of terms produces the above expresion. QED.

An asymptotic expression for f(n)? is now sought. It is seen later that the requirements imposed by the next definitions are sufficient to guarantee that an asymptotic form exists.

**Definition 3**: A factorial generador f(x) for which $f(x) \geqslant 1$, for all $x \geqslant 1$, is called **expandable**.

**Definition 4**: An expandable factorial generator f(x) for which ln f(x) is a concave downward fuction is called **log-concave**.

It can be shown that if $f \in C^2[R]$, a necessary and sufficient condition for f(x) to be log-concave is that

$$f(x)\, f''(x) - (f'(x))^2 \leqslant 0 \ ;$$

in particular, if f(x) is concave downward, the f(x) is log-concave. We begin with a subcase.

**Theorem 1**: Let f(x) be log-concave with f(1) = 1 Then,

$$f(n)\ ? \approx \sqrt{f(n)}\, e^{\sigma(n)}\, e^{\gamma}$$

with

$$\sigma(x) = \int_1^x \ln f(t)\, dt$$

and

$$-\sigma\left(\frac{3}{2}\right) \leqslant \gamma \leqslant 0$$

where ≈ means asymptotically equal.

Proof: Consider

$$a_n = \ln(f(n)!) - 1/2 \ln f(n)$$
$$= \ln f(2) + \ln f(3) + \ldots + \ln f(n-1) + 1/2 \ln f(n),$$

by virtue of the fact that $f(1) = 1$. Consider the curve $y = \ln f(x)$. The area under the curve and between the two lines $x = 1$ and $x = n$ is

$$A = \int_1^n \ln f(x) .$$

This area can be approximated by the sum of the areas of the $n$ trapezoids which are bounded by the lines $x = k-1$ and $x = k$, $k = 2,3\ldots,n$. See Figure 1. The approximated area is

$$1/2(\ln f(1) + \ln f(2)) + 1/2(\ln f(2) + \ln f(3)) + \ldots$$
$$+ 1/2(\ln f(n-1) + \ln f(n)) = \ln f(2) + \ln f(3) + \ldots$$
$$+ \ln f(n-1) + 1/2 \ln f(n) = \ln(f(n)?) - 1/2 \ln f(n)$$
$$= a_n .$$

which is smaller than the exact area, since the region under the curve $y = \ln f(x)$ is convex, by virtue of the fact that $f(x)$ is log-concave. Therefore

$$a_n \leqslant \int_1^n \ln f(x)\, dx \qquad (1)$$

On the other hand, the area under the curve $y=\ln f(x)$ between the lines $x = 3/2$ and $x = n$ is

$$B = \int_{3/2}^n \ln f(x)\, dx ,$$

which can be approximated by the sum of the areas of the (n-1) trapezoids bounded by the tangent at the point (k, ln f(k)) and the lines x = k-1/2 , x = k+1/2 for k = 2,3,...,n-1 , together with the area of the rectangle bounded by the horizontal line at the point (n, ln f(n)) and the two lines x = n-1/2 and x = n. See figure 2. The approximated area is

$$\ln f(2)+\ln f(3)+\ldots+\ln f(n-1)+1/2 \ln f(n) = a_n .$$

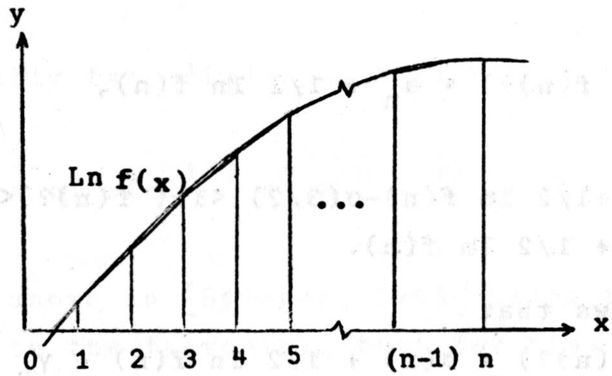


Figure 1

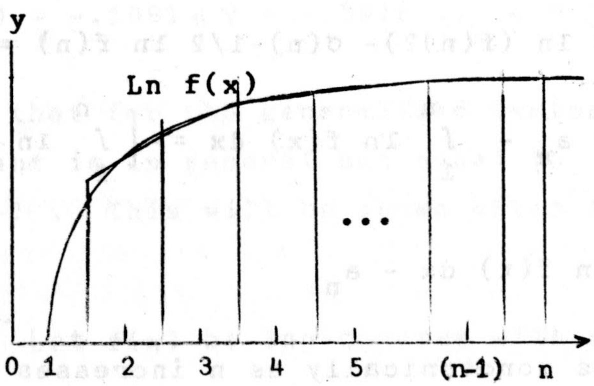


Figure 2

Again,

$$\int_{3/2}^{n} \ln f(x)\, dx \leqslant a_n . \qquad (2)$$

Combining inequalities (1) and (2), we get

$$\int_{3/2}^{n} \ln f(x)\, dx < a_n < \int_{1}^{n} \ln f(x)\, dx.$$

Let

$$\sigma(x) = \int_{1}^{x} \ln f(t)\, dt$$

Then

$$\sigma(n)-\sigma(3/2) < a_n < \sigma\,(n)$$

Since

$$\ln\,(\ f(n)?) = a_n + 1/2 \ln f(n),$$

we have

$$\sigma(n)+1/2 \ln f(n)-\sigma(3/2) < \ln(\ f(n)?) < \sigma(n) + $$
$$+ 1/2 \ln f(n).$$

It follows that

$$\ln(f(n)?) = \sigma(n) + 1/2 \ln f(n) + \gamma_n \ ,$$
$$-\sigma(3/2) \leqslant \gamma_n \leqslant 0 \ .$$

Since

$$\gamma_n = \ln\,(f(n)?) - \sigma(n) - 1/2 \ln f(n) = a_n - \sigma(n)$$

$$= a_n - \int_{1}^{n} \ln f(x)\, dx = -\left[\int_{1}^{n} \ln f(x) dx - a_n\right],$$

and

$$\int_{1}^{n} \ln f(x)\, dx - a_n$$

increases monotonically as n increases (due to the fact that it represents the difference between the area under the curve $y = \ln f(x)$ and the sum of the areas of the trapezoids in Figure 1 we can state $\gamma_n$ decreases monotonically as n increases. However, since $\gamma_n$ has a lower bound of $-\sigma(3/2)$, the sequence of $\gamma_n$ converges by the Bolzano-Weierstrass theorem to a value $\gamma$ with

$$-\sigma(3/2) \leqslant \gamma \leqslant 0 .$$

Using this as an approximation to all the $\gamma_n$, we get

$$\ln (f(n)?) = \sigma(n)+1/2 \ln f(n) + \gamma$$

from which the desired result follows. QED.

Naturally for $f(x) = x$, $\sigma(n) = n \ln n-n+1$, from which

$$f(n)? = n! \approx (n)^{1/2} n^n e^{-n} e^{\gamma+1}$$

It is shown in [Spiegel, 1963] using the Gamma extension to the factorial, that for $f(x) = x$, $e^{\gamma+1} = (2\Pi)^{1/2}$, requiring $\gamma = \ln(\sqrt{2\Pi})-1 = -.0816..$ As the above theorem attests

$$-\sigma(\tfrac{3}{2}) = -.1081 \leqslant \gamma = -.0816... \leqslant 0 .$$

It is clear that for the generalized factorial, this constant is in general not equal to $\ln(2\Pi)^{1/2} -1$. This will be shown after the following

<u>Theorem 2</u>: Let f(x) be log-concave with f(1)>1. Then

$$f(n)? \approx [f(1)]^{n-(1/2)} f(n)^{1/2} e^{\tilde{\sigma}(n)} e^{\tilde{\gamma}}$$

with

$$-\tilde{\sigma}(3/2) \leqslant \tilde{\gamma} \leqslant 0$$

where

$$\tilde{\sigma}(x) = \int_1^x \ln (f(t)/f(1)) \, dt .$$

Proof: Consider $g(x) = f(x)/f(1)$. Then

$$g(n)? = \frac{f(1)}{f(1)} \cdot \frac{f(2)}{f(1)} \cdot \frac{f(3)}{f(1)} \cdots \frac{f(n)}{f(1)} .$$

Hence $f(n)? = (f(1))^n \, g(n)?$ ; consequently

$$f(n)? \approx (f(1))^n \, (g(n))^{1/2} \, e^{\tilde{\sigma}(n)} \, e^{\tilde{\gamma}} =$$

$$= [f(1)]^{n-(1/2)} f(n)^{1/2} \, e^{\tilde{\sigma}(n)} \, e^{\tilde{\gamma}}$$

with

$$\tilde{\sigma}(x) = \int_1^x \ln g(t) \, dt . \quad \text{QED.}$$

Example 1: Consider $f(x) = e^x$. Then

$$f(n)? = e^1 e^2 e^3 \ldots e^n = e^{\sum^n i} = e^{n(n+1)/2}$$

Using the asymptotic expansion,

$$f(n)? \approx e^{n-1/2} \, e^{n/2} \, e^{\int_1^n (t-1)dt} \, e^{\tilde{\gamma}}$$

$$= e^{n-1/2} \, e^{n/2} \, e^{.5n^2-n+.5} \, e^{\tilde{\gamma}}$$

$$= e^{n^2/2+n/2} \, e^{\tilde{\gamma}} = e^{n(n+1)/2} \, e^{\tilde{\gamma}}$$

For this to agree with the exact formula we need $\tilde{\gamma} = 0$. Indeed, computing $\tilde{\sigma}(x)$, we obtain $\tilde{\sigma}(x) = (x^2/2-x+1/2)$, so that $\tilde{\sigma}(3/2) = \frac{1}{8}$ and

$$-\tilde{\sigma}(3/2) \leqslant \tilde{\gamma} \leqslant 0$$

becomes

$$-1/8 \leqslant \tilde{\gamma} \leqslant 0.$$

Observe that $\tilde{\gamma}$ is not $\ln(2\Pi)^{1/2}-1 = -.0816\ldots$ . For this particular case the upper bound for $\tilde{\gamma}$ is achieved. The reason should be evident, since for the function at hand

$$\tilde{\sigma}(x) = \int_1^x \ln \frac{e^t}{e}\, dt = \int_1^x (t-1)dt$$

and the trapezoidal approximation gives the exact answer.

Example 2: Consider $f(x) = ax^p$. Using Proposition 2, we get $f(n)? = a^n(n!)^p$ . Now employing Stirling's formula,

$$f(n)? \approx a^n(2\Pi)^{p/2}\, n^{(n+1/2)p}\, e^{-np}.$$

Carrying out the steps of Theorem 2,

$$f(n)? \approx a^{n-1/2}\, a^{1/2}\, n^{p/2}\, e^{\int_1^x \ln x^p dx}$$

$$= a^n\, n^{p/2}\, e^{p\{n\ln n-n+1\}}\, e^{\tilde{\gamma}}$$

$$= a^n\, n^{p(n+1/2)}\, e^{-np}\, e^p\, e^{\tilde{\gamma}}$$

so that

$$(2\Pi)^{p/2} = e^p\, e^{\tilde{\gamma}}$$

or

$$\tilde{\gamma} = \ln\,(2\Pi)^{p/2} - p$$

Example 3: Consider $f(x) = xe^x$. Clearly

$$f(n)?=n!\; e^{n(n+1)/2} \simeq n^{1/2} n^n\, e^{-n}\, \sqrt{2\Pi}\, e^{\frac{n(n+1)}{2}} .$$

From theorem 2 ,

$$f(n)? \approx e^{n-1/2}(ne^n)^{1/2}\, e^{\int_1^n \ln(\frac{xe^x}{x})dx}\, e^{\tilde{\gamma}}$$

$$= n^{1/2}\, e^{-n}\, n^n\, e^{\frac{n(n+1)}{2}}\, ee^{\tilde{\gamma}}.$$

Thus

$$\tilde{\gamma} = \ln\left(\frac{\sqrt{2\Pi}}{e}\right) ,$$

From the above examples it is clear that $\gamma$ depends on the factorial generator at hand. Using the exact expression for the trapezoidal error, as in [Young, 72] we obtain

<u>Proposition 4</u>: Let $f(x)$ be log-concave, $f(1)=1$ and $f \in C^2[R]$. Then if

$$Q(z) = (f(z)f''(z) - (f'(z))^2)/(f(z)^2),$$

$$(1)\gamma_n = (n-1)\, Q(\varepsilon)/12 \quad \text{where} \quad 1<\varepsilon<n.$$

$$(2)\gamma_n \leqslant (n-1) \max_{1\leqslant z\leqslant n} Q(z)/12 .$$

This formulation is, however, not too useful since it does not show that $\gamma_n$ converges to a limit. Such convergency could be established if one could for example prove that $n/2\leqslant\varepsilon\leqslant n$.

The situation is remedied by the next theorem.

<u>Theorem 3</u> : Let $f(1) = 1$ then

$$\gamma = -\frac{1}{12}\,\frac{f'}{f} - \frac{1}{2}\int_1^\infty \overline{B}_2(x)\, \frac{f''\, f-f'^2}{f^2}\, dx$$

where $\overline{B}_2(x)$ is the modifield Bernoulli polynomial of degree 2.

Proof: Let $B_n(x)$ be the n-th degree Bernoulli polynomial; let $\overline{B}_n(x) = B_n(x-[x])$. Then [Abramowitx, 1964] shows that the Euler-MacLaurin Sum Formula is

$$\sum_{k=0}^{m-1} F(a+kh+wh) = \frac{1}{h} \int_a^b F(t)\,dt +$$

$$+ \sum_{k=1}^{p} \frac{h^{k-1}}{k!} B_k(w) \{F^{(k-1)}(b) - F^{(k-1)}(a)\} -$$

$$- \frac{h^p}{p!} \int_0^1 \overline{B}_p(w-t) \{ \sum_{k=0}^{m-1} F^{(p)}(a+kh+th)\}\,dt$$

$$p \leqslant 2n, \quad 1 \geqslant w \geqslant 0$$

where the coefficients $b_k$ of the Bernoulli polynomials $B_n(x) = \sum_{k=0}^{n} b_k x^k$ are

| n/k | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | | | | |
| 1 | -1/2 | 1 | | | |
| 2 | 1/6 | -1 | 1 | | |
| 3 | 0 | 1/2 | -3/2 | 1 | |
| 4 | -1/30 | 0 | 1 | -2 | 1 |

Evaluating this for $F(x) = \ln f(x)$, $f(1) - 1$ , $f(x) \varepsilon C^2$,

$$\sum_{m=1}^{n} \ln f(m) = 1/2(\ln f(1) + \ln f(n)) + \int_1^n \ln f(x)\,dx +$$

$$+ \frac{1}{12} \left( \frac{f'}{f}(n) - \frac{f'}{f}(1) \right) - 1/2 \int_1^n \overline{B}_2(x) \frac{f''f-f'^2}{f^2} dx =$$

$$= \ln \sqrt{f}(n) + \sigma(n) - 1/2 \frac{f'}{f}(1) - 1/2 \int_1^\infty \overline{B}_2(x) \frac{f''f-f'^2}{f^2} dx +$$

$$+ \frac{1}{12} \frac{f'}{f}(n) + 1/2 \int_n^\infty \overline{B}_2(x) \frac{f''f-f'^2}{f^2} dx ,$$

where use has been made of $f(1) = 1$ ,

Consequently, $f(n)?=\sqrt{f(n)}\, e^{\sigma(n)} e^{\gamma} e^{\varepsilon}$ with

$$\gamma = - \frac{1}{12} \frac{f'}{f}(1) - 1/2 \int_1^\infty \overline{B}_2(x) \frac{f''f-f'^2}{f^2} dx$$

and

$$\varepsilon = \frac{1}{12} \frac{f'}{f}(n) + 1/2 \int_n^\infty \overline{B}_2(x) \frac{f''f-f'^2}{f^2} dx ,$$

assuming that the infinite integral converges ( for this the log-concavity of f is a sufficient, but not necessary condition). If we now assume in addition that $\frac{f'}{f}(x) \to 0$ for $x \to \infty$ , (weaker than log-concavity), then $\varepsilon \to 0$ for $x \to \infty$ and the asymptotic relation follows. QED.

In general, however, it is not easy to evaluate $\gamma$ exactly; thus one must be content with the bound

$$- \sigma\left(\frac{3}{2}\right) \leq \gamma \leq 0.$$

Actually $-\sigma\left(\frac{3}{2}\right)$ is the best bound one can get for a general f(x); for a specific f, such bound can be improved as follows.

Theorem 4 : Let $1 < a \leqslant \frac{3}{2}$ be such that

$$(a-\frac{3}{2}) \ln f(a)+ \int_a^{\infty} \overline{B}_1(x) \frac{f'}{f} (x)\, dx \geqslant 0.$$

Then

$$-\sigma(a) \leqslant \gamma \leqslant 0$$

Proof: Note first of all that if $f(x) = c > 1$ then the condition reduces to

$$(a-\frac{3}{2}) \ln c \geqslant 0$$

thus $a = \frac{3}{2}$ is the tightest general bound. By the preceding formula for $\gamma$, one has to show that

$$\sigma(a) \geqslant \frac{1}{12} \frac{f'}{f} (1)+ \frac{1}{2} \int_1^{\infty} \overline{B}_2(x) \frac{f''f-f'^2}{f^2} dx \geqslant 0.$$

The last inequality is immediate: From $f''f-f'^2 \leqslant 0$ it follows that we minimize the last integral, if we replace $\overline{B}_2(x)$, by its largest positive value, namely 1/6, and obtain

$$\frac{1}{12}(\frac{f'}{f} (1)+ \int_1^{\infty} d(\frac{f'}{f}))= \frac{1}{12} (\frac{f'}{f}(1)-\frac{f'}{f}(1)+\frac{f'}{f}(\infty)) = 0$$

The first one is equivalent to

$$\int_1^a \ln f(x)dx \geqslant \frac{1}{12} \frac{f'}{f}(1)+\frac{1}{2} \int_1^{\infty} \overline{B}_2 (x)\, d(\frac{f'}{f}) .$$

Integrating by parts, the second member equals

$$\frac{1}{12} \frac{f'}{f}(1)+\frac{1}{2} \overline{B}_2(x)\frac{f'}{f}(x)\Big]_1^{\infty} -\frac{1}{2} \int_1^{\infty} \frac{f'}{f} \circ 2 \overline{B}_1(x)\, dx =$$

$$= - \int_1^{\infty} \overline{B}_1 (x) \frac{f'}{f}(x)\, dx \circ$$

The first member equals

$$x \ln f(x) \Big]_1^a - \int_1^a x \frac{f'}{f}(x)dx = a \ln f(3/2) -$$

$$- \int_1^a (x-[x]-\tfrac{1}{2})\frac{f'}{f}(x)dx - \frac{3}{2}\int_1^a \frac{f'}{f}(x)\, dx$$

where the 3/2 comes about because on the interval $[1,a]$ , $a \leqslant 3/2 - [x] - 1/2 = - 3/2$. But this expression equals

$$(a-3/2)\ln f(a)+ \int_1^a \overline{B}_1(x) \frac{f'}{f}(x)\, dx$$

Thus we need

$$(a-3/2) \ln f(a)+ \int_a^\infty \overline{B}_1(x) \frac{f'}{f}(x)\, dx \geqslant 0.$$

This proves the theorem. QED.

If $a = 3/2$, this condition holds. Indeed, in each interval $n - 1/2$, $\overline{B}_1(x) = x-[x]- 1/2$ varies linearly from 0 to 1/2 and from -1/2 to 0, while $\frac{f'}{f}(x) \geqslant 0$, but decreases (this is equivalent to the log of $f(x)$); hence,

$$\int_{n-1/2}^{n+1/2} \overline{B}_1(x) \frac{f'}{f}(x)\, dx \geqslant 0.$$

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