Revista Colombiana de Matemáticas
Vol. XI (1977), págs. 113 - 119

INTERSECTIONS OF NORMALLY CONTAINED FITTING CLASSES

by

Michael A. KLEMBARA and Donald B. PARKER

ABSTRACT

A theorem of Blessenohl and Gaschütz is generalized to show that the intersection of a family of Fitting classes, each of which is normal in some Fischer class $\mathcal{Y}$, is normal in $\mathcal{Y}$.

All groups considered in this paper will be finite and solvable. We will let the class all finite solvable groups be denoted by S. A class F of groups is a Fitting class if:

(i) $G \in F$, $N \triangleleft G$ implies $N \in F$, and

(ii) $N, M \triangleleft G$, $N, M \in F$ implies $NM \in F$.

A **Fischer class** is a Fitting class F, which also satisfies:

(iii) if $G \in F$, $H \leqslant G$ and $H/\mathrm{core}_G(H) \in N$, the class of nilpotent groups, then $H \in F$.

It follows from (ii) that in any group G, there exists a unique normal subgroup which is maximal with respect to belonging to a Fitting class F. We call this subgroup the **F radical** of G and denote it G_F. We note that G_F contains every subnormal F-subgroup of G. A subgroup V of G is called an F- injector of G if $V \cap M$ is F-maximal in M for every subnormal subgroup M of G. In [3], it is shown that for an arbitrary Fitting class F, each group G has a unique conjugacy class of F-injectors. Basic properties of Fitting classes and F-injectors are found in [3] and [4] .

In [1], Blessenohl and Gaschütz defined a **normal Fitting class** to be one for which the F-radical is F-maximal in G for every G in S. Since the F-radical is contained in each F-injector for an arbitrary Fitting class F, when F is a normal Fitting class there is only one F-injector, namely the F-radical. The following theorem is proved in [1].

Theorem 1. The intersection of any collection of normal Fitting classes is again a normal Fitting class.

Cossey, in [2] , extends the definition of normal Fitting classes as follows:

Definition 2. Let F and Y be two Fitting classes such that $F \subseteq Y$. We say that F is normal in Y if G_F is F-maximal in G for every $G \in Y$. We denote this $F \triangleleft Y$, and remark that if Y is S then we have the usual definition of a normal Fitting class. The following well known proposition gives an example of definition 2.

Proposition 3. If F is a Fitting class then $F \triangleleft FN$, where

$$FN = \{G \mid G/G_F \in N\}.$$

Proof: Let $G \in FN$, then $G/G_F \in N$. Let V be an F-injector of G, then V/G_F is subnormal in G/G_F. Hence V is subnormal in G and so V must be in G_F.

In this note we generalize Theorem 1 to include Cossey's definition of normality within a given Fitting class Y. For this we need the following:

Lemma 4. [5, p. 568] Let A, B and C be subgroups of G. Then the following statements are equivalent:

(a) $A \cap BC = (A \cap B)(A \cap C)$.

(b) $AB \cap AC = A(B \cap C)$.

Theorem 5. Let I be an indexing set. If for each $i \in I$, F_i is a Fitting class such that $F_i \triangleleft Y$, Y a Fischer class, then $\bigcap_{i\in I} F_i = F \triangleleft Y$.

Proof: The proof is by induction on the order of groups in Y. Let G be a group of minimal order in Y such that there exists an F-injector, say V, which is not normal in G. Let V_i be the F_i-injectors of G for each $i \in I$. Then $V_i \triangleleft G$ for each $i \in I$, by hypothesis. Hence $\bigcap_{i\in I} V_i \triangleleft G$ and $\bigcap_{i\in I} V_i \in F$. Therefore $\bigcap_{i\in I} V_i \leq G_F$. But $G_F \leq G_{F_i} = V_i$ for each $i \in I$, so $G_F = \bigcap_{i\in I} V_i$.

Since V is not normal in G, we have $\bigcap_{i\in I} V_i \nleq V$.

Let M be a maximal normal subgroup of G.

(a) $V \cap M = (\bigcap_{i\in I} V_i) \cap M$. For by induction $V \cap M \triangleleft M$, since $M \in Y$ and $V \cap M$ is an F-injector of M. Hence $V \cap M = M_F = G_F \cap M = (\bigcap_{i\in I} V_i) \cap M$.

(b) V is not contained in any proper normal subgroup of G. For if $V \leq N \triangleleft G$, then by the induction hypothesis $V \triangleleft N$. Hence V is subnormal in G and $V = G_F$ a contradiction.

(c) $G = NV$ where N is any normal subgroup of G, such that $G/N \in N$. For suppose $NV \neq G$, then NV/N

would be subnormal in G/N, hence NV would be subnormal in G. But then there would exist a proper normal subgroup H of G, such that $V \leq H$ a contradiction of (b).

(d) M is the unique maximal normal subgroup of G. For, if not, let M_1 and M_2 be two maximal normal subgroups of G. We have $G/M_1 \in N$ and $G/M_2 \in N$ since maximal normal subgroups have prime index. By [5, Satz 2.5, p. 261], we have $G/M_1 \cap M_2 \in N$. So, by (c) we have $G = M_1V = M_2V = (M_1 \cap M_2)V$. Thus $G = M_1V \cap M_2V = (M_1 \cap M_2)V$ and we can apply Lemma 4 to obtain $V = V \cap (M_1 M_2) = (V \cap M_1)(V \cap M_2)$. However, by part (a) of this proof $(V \cap M_1) = (\bigcap_{i\in I} V_i) \cap M_1$ and $(V \cap M_2) = (\bigcap_{i\in I} V_i) \cap M_2$. So $V = ((\bigcap_{i\in I} V_i) \cap M_1)((\bigcap_{i\in I} V_i) \cap M_2) \leq (\bigcap_{i\in I} V_i)$, a contradiction. Therefore $V_i \leq M$ for all $i \in I$ and $V \cap M = (\bigcap_{i\in I} V_i) \cap M = \bigcap_{i\in I} V_i$. So $G/M = MV/M \cong V/V \cap M = V/(\bigcap_{i\in I} V_i)$. Hence $|V : (\bigcap_{i\in I} V_i)| = p$, for some prime p.

(e) V_iV is properly contained in G for some $i \in I$. For if not, there then exists some $j \in I$ such that $V_j \not\leq G$, otherwise $G \in F$ a contradiction. But then $G/V_j = V_j V/V_j = V/V_j \cap V$ and so V_j must have index p in G. Hence $V_j = M$ by (d). We now claim that $V_j \in F = \bigcap_{i\in I} F_i$. If $i \in I$ is such that $V_i \not\leq G$, then by the above argument $V_i = M = V_j$, so $V_j \in F_i$. If $i \in I$ is such that $V_i = G$ then $V_j \triangleleft G$ implies $V_j \in \mathcal{F}_i$. Hence $V_j \in \bigcap_{i\in I} \mathcal{F}_i = \mathcal{F}$. But then $V_j \leqslant G_{\mathcal{F}} < V$, which implies $M \leqslant V$ a contradiction. Thus there exist some $i\varepsilon I$ such that $V_iV \nleqslant G$.

(f) If $V_iV \nleqslant G$ then $V_iV \in \mathcal{Y}$. Here we utilize the fact that $\mathcal{Y}$ is a Fischer class. We have $V_iV/V_i \cong V/V_1 \cap V \in \mathcal{S}_p$ the class of finite solvable p-groups. Also $V_i \triangleleft G$ implies $V_i \in \mathcal{Y}$ and $V_i \leqslant \mathrm{core}_G(V_1V)$. Hence $V_iV/\mathrm{core}_G(V_iV) \in \mathcal{S}_p \subsetneq \mathcal{N}$ implies $V_iV \in \mathcal{Y}$.

Finally, since V is an $\mathcal{F}$-injector of V_iV and by the induction hypothesis, we have V normal in V_iV. But $V \in \mathcal{F}_i$ and V_i is the $\mathcal{F}_i$-injector of V_iV, so $V \leqslant V_i$ a contradiction of (b) and the theorem is proven.

***

REFERENCES

[1] Blessenohl, D. and Gaschutz, W. Uber normale Schunck und Fittingklassen , Math. Zeit. 118, (1970), 1-8.

[2] Cossey, J. Products of Fitting Classes. Math. Zeit. 141 (1975), 289-295.

[3] Fischer, B., Gaschutz, W. and Hartley, B. Injecktoren endlicher auflosbarer Gruppen, Math.Zeit. 102, (1967), 337-339.

[4] Hartley, B. On Fischer's dualization of formation theory, Proc. London Math. Soc. (3), 19, (1969), 193-207.

[5] Huppert, B. Endliche Gruppen I, Berlin: Springer-Verlag (1967).

Northern Kentucky University
Kighland Heights, Kentucky 41076

University of Cincinnati
Cincinnati, Ohio 45221

(Recibido en junio de 1976).