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SEMINILPOTENT GROUPS

por

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Introduction.

In this paper, we initiate the study of a class of groups which contains the classes of nilpotent and perfect groups and which seems to have gone unnoticed in the literature. We announce here some elementary properties of these groups, leaving the study of other important properties to the future.

§ 1. Basic notations and results used.

In this section, we recall some well known results from the literature on products, factor groups and commutator subgroups, for our use. Before this, we explain some basic notations.

The subgroup $[A,B]$, for subgroups A and B of a group G, is the subgroup generated by the elements

$$a^{-1} b^{-1} ab = [a,b], \quad \text{for all } a \in A \text{ and } b \in B.$$

If $A \Delta G$, then $[A,B] \subseteq A$. If A and B are both normal (characteristic) subgroups then $[A,B]$ is a normal (characteristic) subgroup of G. In particular, the characteristic subgroup $[G,G] = G'$ is called the derived group of G; G/G' is abelian, and if G/A is abelian for a subgroup $A \Delta G$ then $G' \subseteq A$.

A group G is the direct product of a set $\{G_i\}_{i \in I}$ of its subgroup G_i, for an index set I, if the following conditions are satisfied:

1. $G_i \Delta G$, for all $i \in I$,
2. $G = \langle \bigcup_{i \in I} G_i \rangle$,
3. $G_i \cap \langle \bigcup_{j \neq i} G_j \rangle = 1$, for each $i \in I$.

The direct product will be denoted by $\otimes_I G_i$ and the G_i will be called the direct factors.

We denote the cartesian product of a family $\{G_i\}_{i \in I}$ of groups by $^*\prod_I G_i$, and consider the subset of this group consisting of all elements in the underlying cartesian set product $\times_I G_i$ having at most a finite number of components different from the identity elements of the G_i. It is easily shown this set to be a subgroup of $^*\prod_I G_i$

and will be denoted by $\prod_I G_i$.

Next we list some well known results, whose proofs are either easy or avaible in the literature.

(1.1) Let G be a direct product of its subgroups $\{G_i\}_I$. Then there is a unique isomorphism

$$\eta \in \mathrm{Hom}\,(\otimes_I G_i\ ,\ \prod_I G_i\ )$$

(1.2) If $A \Delta G$, and $\{G_i\}_I$ is a family of subgroups of G, each containing A, then

$$\bigcap_i \left(G_i\ /\ A\right) = \left(\bigcap_i G_i\right) / A\ .$$

(1.3) Let $G = \otimes_I G_i$, $H = \otimes_I H_i$ then

$$[G,\ H] = \otimes_I [G_i,\ H_i]$$

(1.4) If A, B Δ G, then

$$(AB)' = [AB,\ AB] = A'[\ A, B\ ]\ B'\ .$$

(1.5) If $K \Delta G$ and $K \subseteq H \subseteq G$, then

$$[G/K,\ H/K] = [G, H]\ K/K.$$

In particular, $[H/K,\ H/K] = H'\ K/K$.

We recall that a group G is called perfect if it coincides with its derived group G' .

Clearly, a perfect group has no nontrivial abelian factor group. A property of perfect group recorded in [2] is

(1.6) Every epimorphic image of a perfect group is perfect. For many results in this section the reader is referred to [1], [2] .

§ 2. Seminilpotent groups.

In a group G, we have the well known constructions of the following (possibly transfinite) chain of subgroups:

(A) Its lower central chain

$$G = D_o(G) \supseteq D_1(G) \supseteq \ldots \supseteq D_i(G) \supseteq D_{i+1}(G) \supseteq \ldots$$

where $D_{i+1}(G) = [D_i(G), G]$, and

(B) its derived chain

$$G = P_o(G) \supseteq P_1(G) \supseteq \ldots \supseteq P_i(G) \supseteq P_{i+1}(G) \supseteq \ldots$$

where $P_{i+1}(G) = [P_i(G), P_i(G)]$.

We notice that each term of the lower central chain and of the derived chain are determined by variety functors in category theoretic language (cf [3], [4]); as such, they are characteristic subgroups.

We define $D(G) = \bigcap_i D_i(G)$ and $P(G) = \bigcap_i P_i(G)$, and notice that $P(G) \subseteq D(G)$ for any group G.

It was shown by Finkelstein [2] that P(G) is the unique maximal perfect normal subgroup of G, and that P(G) is a radical.

Though $D(G)$ is commonly called a hypercommutator (subgroup) (cf. Bechtell 1, page. 49), it is in fact a radical like $P(G)$. We prove this by showing that $G/D(G)$ is semi simple, i.e.

<u>Theorem</u> 2.1. For any group $G, D(G/D(G)) = 1$.

<u>Proof</u>: Let $K = G/D(G)$. We will check by induction that $D_i(K) = D_i(G) / D(G)$. We have $D_o(K) = G/D(G) = D_o(G) / D(G)$. Assume we already have $D_i(K) = D_i(G) / D(G)$.

Then. $D_{i+1}(K) = [D_i(K), D_o(K)] =$

$$= [D_i(G)/D(G), D_o(G)/D(G)] \frac{[D_i(G), D_o(G)]\, D(G)}{D(G)},$$

by (1.5). Since $D(G) \subseteq D_{i+1}(G)$,

$$\frac{D_{i+1}(G)\, D(G)}{D(G)} = \frac{D_{i+1}(G)}{D(G)}$$

Therefore, $D_i(K) = D_i(G)/D(G)$. Hence,

$$D(K) = \bigcap_i D_i(K) = \bigcap_i D_i(G)/D(G) = \frac{\bigcap_i D_i(G)}{D(G)} =$$

$$= D(G)/D(G) = 1. \quad \text{(by (1.2))}.$$

It is immediate that $P(G/D(G)) = 1$ and that $D(G/P(G)) = D(G)/P(G)$.

<u>Definition</u>. 2.2. We call a group G seminilpotent if $P(G) = D(G)$ i.e., when $D(G) \subseteq P(G)$.

<u>Examples</u>: 1. All nilpotent groups are seminilpo-

tent, since $P(G) = D(G) = 1$.

2. All perfect groups are seminilpotent.

3. Solvable groups are not necessarily seminilpotent.

For example, the permutation group S_3, here $P(S_3) = 1$, but $D(S_3) = A_3$.

<u>Proposition</u> 2.3. If, for a group G, a term $D_i(G)$ of the lower central chain is perfect, then the lower central chain becomes stationary at $D_i(G)$ and G is seminilpotent with.

$$P(G) = D(G) = D_i(G).$$

<u>Proof</u>: From the construction of the lower central chain $\ldots D_i(G) \supseteq D_{i+1}(G) \supseteq \ldots$, $D_i(G)$ perfect implies

$$D_i(G) = [D_i(G), D_i(G)] \subseteq [D_i(G), G] = D_{i+1}(G),$$

and therefore the lower central chain becomes

$$\ldots \supseteq D_i(G) = D_{i+1}(G) = \ldots .$$

Now, $D(G) = \bigcap_i D_i(G) = D_i(G)$,

and $P(G) \subseteq D(G) = D_i(G)$.

Since $P(G)$ is the unique maximal perfect normal subgroup of G by Finkelstein theorem [2], we have $P(G) = D_i(G) = D(G)$.

We remark here the fact that $D(G)$ is the term $D_i(G)$ of the lower central chain where the chain becomes stationary.

Proposition 2.4. For a group $G, D(G) = P(G)$ implies that $D(G)$ is the maximal perfect normal subgroup of each $D_i(G)$.

Proof: Since $D(G) \subseteq D_i(G)$, we have $P[D(G)] \subseteq P[D_i(G)]$, i.e., $D(G) \subseteq P[D_i(G)]$. Also, $D_i(G) \subseteq G$ implies $P[D_i(G)] \subseteq P(G) = D(G)$. Therefore, $D(G) = P[D_i(G)]$.

Proposition 2.5. For a finite group G, $G/P(G)$ is nilpotent if and only if G is seminilpotent.

Proof: For a finite group G having hypercommutator subgruop $D(G)$, $G/D(G)$ is nilpotent, and this is the unique characteristic subgroup having this and the additional property that if G/A is nilpotent then $D(G) \subseteq A$. Thus $G/P(G)$ nilpotent implies $D(G) \subseteq P(G)$. The second part is immediate.

Remark. 2.6. By looking at the example of nilpotent and perfect groups, it is to be expected that the epimorphic image of a seminilpotent group is seminilpotent. However, this is not true. This is so because the free groups, being residually nilpotent and residually solvable, are certainly seminilpotent, but their epimorphic images need not to be so. An example is the free group of rank 2 whose epimorphic image is the symmetric group S_3. [c.f. page 205, 8.4.16, Scott [6]].

Lemma. 2.7. For a group G,

$$\bigcap_i \prod_t D_i(G_t) = \prod_t \bigcap_i D_i(G_t)$$

Proof. Let $A = \bigcap_i \prod_t D_i(G_t)$ and $B = \prod_t \bigcap_i D_i(G_t)$.

If $x = (x_t) \in B$, then, for all t, $x_t \in \bigcap_i D_i(G_t)$.

Therefore, for all t and i, $x_t \in D_i(G_t)$, i.e., $(x_t) \in \prod_t D_i(G_t)$ for all i. Hence $(x_t) \in A$, and so $B \subseteq A$.

Again, if $x = (x_t) \in A$, then $(x_t) \in \prod_t D_i(G_t)$ for all i. Therefore, for all i and t, $x_t \in D_i(G_t)$, i.e., $x_t \in \bigcap_i D_i(G_t)$ for any t. So $(x_t) \in B$, and $A \subseteq B$.

Proposition 2.8. If $\{G_i\}_I$ is a family of groups, each one being seminilpotent, then so is $\prod_I G_i$.

Proof: We check by induction

$$D_t(\prod_I G_i) = \prod_I D_t(G_i).$$

We notice that $D_o(\prod_I G_i) = \prod_I G_i = \prod_I D_o(G_i)$ if we already had $D_t(\prod_I G_i) = \prod_I D_t(G_i)$, then $D_{t+1}(\prod_I G_i) = [D_t(\prod_I G_i), D_o(\prod_I G_i)] = [\prod_I D_t(G_i), \prod_I D_o(G_i)] = \prod_I [D_t(G_i), D_o(G_i)]$, which, by (1.3), $= \prod_I D_{t+1}(G_i)$.

Now $D(\prod_I G_i) = \bigcap_t D_t(\prod_I G_i) = \bigcap_t \prod_I D_t(G_i)$ by above, And by Lemma (2.7), this is further equal to $\prod_I \bigcap_t D_t(G_i) = \prod P_i$, where each $P_i = \bigcap_t D_t(G_i)$ is a perfect group, since each G_i is seminilpotent. Now $\prod P_i$ is perfect, as can be easily seen from the Lemma below:

Lemma 2.9. If $G = \otimes_I G_i$ and each G_i is perfect, then G is perfect.

Proof: $[G, G] = [\otimes_I G_i, \otimes_I G_i]$. But, by (1.3),

$$[G, G] = \otimes_I [G_i, G_i] = \otimes_I G_i = G$$

To complete the proof of proposition 2.8., we notice that $D(\prod_I G_i)$ is perfect by Lemma 2.9. Hence, $\prod_I G_i$ is seminilpotent by (1.1).

Proposition 2.10. The extension of a perfect group by a seminilpotent group is seminilpotent.

Proof: Let G be the extension of a perfect group A by a seminilpotent group B. We notice that $B \cong G/A$ is seminilpotent and A is perfect.

We now use induction to show that $D_i(B) = D_i(G)/A$. Obviously $D_o(B) = G/A = D_o(G)/A$.

If A is perfect then $A = D_1(A) \subseteq D_1(G)$, and so $D_1(B) = [D_o(B), D_o(B)] = [G/A, G/A] = \dfrac{D_1(G)A}{A}$ by (1.5). Since A in perfect, this is equal to $D_1(G)/A$.

In general, if we already had $D_i(B) = D_i(G)/A$, then

$$D_{i+1}(B) = [D_i(B), D_o(B)] = \left[\frac{D_i(G)}{A}, \frac{G}{A}\right] = \frac{D_{i+1}(G)A}{A}.$$

That $A \subseteq D_{i+1}(G)$, can also be proved by induction, and so $D_{i+1}(B) = \dfrac{D_{i+1}(G)}{A}$.

Now $D(B) = \bigcap_i D_i(B) = \bigcap_i \frac{D_i(G)}{A} = \frac{\bigcap_i D_i(G)}{A}$ by (1.2). Similarly $P(B) = \frac{\bigcap_i P_i(G)}{A}$.

But B seminilpotent implies $\frac{\bigcap_i D(G)}{A} \subseteq \frac{\bigcap_i P_i(G)}{A}$,

i.e., $\bigcap_i D_i(G) \subseteq \bigcap_i P_i(G)$. Thus $D(G) \subseteq P(G)$.
Therefore, G is seminilpotent.

Corollary 2.11. If A is perfect and B is seminilpotent, then A ≀ B is seminilpotent.

Proof: For the definition of A ≀ B, see Scott [6], page 215, and also prop. 9.2.4 and 9.2.5.

Now, A ≀ B has a normal subgroup $G = \Sigma_E\{A|b \in B\}$ such that A ≀ B/G is isomorphic to B. But A perfect implies $\Sigma_E\{A|b \in B\}$ is perfect. (See either Lemma 2.9 above or Ex 3.4.13 of Scott [6]). Thus, A ≀ B is the extension of G by B and, as such, seminilpotent by the above proposition.

§ 3. Remarks.

(1) In a future paper we have the purpose of examining other important properties of seminilpotent groups and investigate the connection of this class of groups with other classes, like the supersolvable ones.

(2) The generalization of these results in category theoretic terms similar to those in [3], [5] is also possible.

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