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SPECIAL ARITHMETIC AND GEOMETRIC MEANS PRESERVE Φ-LIKE UNIVALENCE

by

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Let R be a region containing 0. Let Φ be analytic in R and satisfying $\Phi(0) = 0$ and $\text{Re}\{\Phi'(0)\} > 0$. Let D be the open unit disc of the complex plane centered at 0. Define $S(\Phi)$ as the set of normalized functions, $f(z) = z+a_2a^2+ \ldots$, analytic in D such that $f(D) \subset R$ and

$$\text{(1)} \qquad \text{Re}\left\{\frac{zf'(z)}{\Phi(f(z))}\right\} > 0$$

for all $z \in D$. The elemts of $S(\Phi)$ are called Φ-like in D. Geometrically, we define R to be Φ-like if for any $\gamma \in R$ the initial value problem

$$(2) \qquad \frac{dW}{dt} = -\Phi(W), \quad W(0) = \gamma$$

ha a solution $W(t)$, defined for all $t \geqslant 0$, such that $W(t) \in R$ for all t and $W(t) \to 0$ as $t \to \infty$ With these definitions, Professor Luois Brichkman [1] proved the following two theorems, stated below without proof, together with a corollary.

Theorem A. Let f be Φ-like in D. Then f is univalent in D and f(D) is Φ-like.

Theorem B. Let f be analytic in D with $f(0) = 0$. If f is univalent and f(D) is Φ-like then f is Φ-like in D.

Corollary A. Let f be analytic in D with $f(0) = 0$. Then f is univalent in D if and only if f is Φ-like for some Φ.

With R and Φ as defined above, we define $M(a,b,\Phi(f))$ to be the class of those functions $f(z) = z+a_2 z^2+\ldots$, analytic in D and satisfying $\mathrm{Re}\{K(a,b,\Phi(f))\} > 0$ (a and b real numbers), $f'(z)\Phi(f(z)) \neq 0$ in $0 < |z| < 1$, and also

$$(3) \qquad K(a,b,\Phi(f)) = aA(f,\Phi)+bB(f,\Phi)$$

$$(4) \qquad A(f,\Phi) = 1 + zf''(z)/f'(z) -$$

$$- z(\Phi(f(z)))'/\Phi(f(z))$$

(5) $$B(f,\Phi) = zf'(z)/\Phi(f(z)) .$$

We define $G(a,b,\Phi(f))$ to be the class of analytic functions $f(z) = z+a_2z^2+\ldots$, in D which satisfy $\mathrm{Re}\{T(a,b,\Phi(f))\} > 0$, $f'(z)\Phi(f(z)) \neq 0$ in $0 < |z| < 1$, for a and b real number a+b an odd integer, where :

(6) $$T(a,b,\Phi(f)) = (A(f,\Phi)^a(B(f,\Phi))^b$$

is defined by taking principal branches.

Clearly $M(a,b,\Phi(f))$ and $T(a,b,\Phi(f))$ contain arithmetic and geometric means of the functions $A(f,\Phi)$ and $B(f,\Phi)$ relative to masses a and b, respectively. In this note we demonstrate the following:

<u>Theorem 1</u>. All functions belonging to $M(a,b,\Phi(f))$ or $G(a,b,\Phi(f))$ are Φ-like univalent functions from the class $S(\Phi)$.

<u>Proof</u>. First of all we note if $f \in G(a,b,\Phi(f))$ or $f \in M(a,b,\Phi(f))$ then $\Phi(f)$ is analytic in D and $\Phi(f)$ has no zero in $0 < |z| < 1$. If we define $w(z)$ by the equation

(7) $$\frac{zf'(z)}{(f(z))} = \alpha \left\{ \frac{(1-w(z))}{(1+w(z))} \right\} + i\beta$$

$$= \frac{\delta - \bar{\delta} w(z)}{(1+w(z))} ,$$

where $\delta = (\Phi'(0))^{-1} = \alpha + i\,\beta$, and α and β are real numbers, we find that $w(z)$ is certainly analytic in the neighbourhood of zero. Also, since $f'(z)\,\Phi(f(z)) \neq 0$ in $0 < |z| < 1$, we find that $zf'(z)/\Phi(f(z))$ is analytic in D. Hence, without loss of generality, we may choose $w(z)$ to be regular in D. Also equation (7) implies that $w(0) = 0$. Since $\alpha > 0$, to show that $f \in S(\Phi)$ it is enough to show that $|w(z)| < 1$ for $z \in D$. Suppose this were false. Let $M(r,w) = \max \{|w(z)| : |z| = r\}$, then there is some r_1 such that $M(r_1,w) = 1$, and so there is some $z_1 \in D$ such that $|w(z_1)| = 1$ and $|z_1| = r$. By Jack's lemma there exists $t \geqslant 1$ such that $z_1 w'(z_1) = tw(z_2)$ [2]. Now we compute $A(f,\Phi)$ and $B(f,\Phi)$ from (7) and find that

$$(8) \qquad A(f,\Phi) = -\frac{\bar{\delta}\, zw'(z)}{\delta - \bar{\delta} w(z)} - \frac{zw'(z)}{1+w(z)}$$

$$(9) \qquad B(f,\Phi) = \frac{\delta - \bar{\delta}\, w(z)}{1 + w(z)}$$

From (8) and (9) it follows :

$$(10) \qquad K(a,b,\Phi(f)) = -\frac{a\bar{\delta} zw'(z)}{\delta - \bar{\delta} w(z)} - \frac{azw'(z)}{1+w(z)} +$$

$$+ \frac{b(\delta-\overline{\delta}w(z))}{1+w(z)}$$

$$(11) \quad T(a,b,\Phi(f)) = \left(- \frac{z\overline{\delta}w'(z)}{\delta-\overline{\delta}w(z)} - \frac{zw'(z)}{1+w(z)} \right)^a \cdot \left(\frac{\delta-\overline{\delta}w(z)}{1+w(z)} \right)^b$$

If we require f to be in $M(a,b,\Phi(f))$ and use (7) with $z = z_1$, we find that $|w(z_1)| = 1$ and

$$(12) \quad \operatorname{Re}(K(a,b, (f)))_{\text{at } z = z_1}$$

$$= \operatorname{Re}\left\{- \frac{a\overline{\delta}tw(z_1)}{\delta-\delta w(z_1)} - \frac{atw(z_1)}{1+w(z_1)}\right\}$$

$$+ \operatorname{Re}\left\{ \frac{b(\delta-\overline{\delta}w(z_1))}{1+w(z_1)} \right\}$$

$$= \operatorname{Re}\left(\frac{4ai\alpha(t\beta+t\operatorname{Im}(\delta\overline{w(z_1)}\ )}{|(1+w(z_1))(\delta-\overline{\delta}w(z_1))|^2} \right)$$

$$+ \operatorname{Re}\left(\frac{2ib(\beta+ \operatorname{Im}(\overline{\delta}w(z_1))}{|(1+w(z_1))^2\ |} \right) = 0$$

But this contradicts the fact that $f \in M(a,b,\Phi(f))$. So $|w(z)| < 1$ for all z in D and, from (1), we conclude $f \in S(\Phi)$. Similary, if we require $f \in G(a,b,\Phi(f))$ we have that if $a+b$ is an odd integer then

$$(13) \qquad \text{Re } (T(a,b,\Phi(f)))_{\text{at } z=z_1}$$

$$= \text{Re}((A(f,\Phi))^a (B(f,\Phi))^b)_{\text{at } z=z_1}$$

$$= \text{Re} \left\{ \left(\frac{4\alpha t_1(\beta+\text{Im}(\delta \overline{w(z_1)}))i}{|(1+w(z_1))(\delta-\overline{\delta}w(z_1)))|^2} \right)^a \right.$$

$$\left. \cdot \left(\frac{2(\beta+\text{Im}\delta w(z_1))^i}{|(1+w(z_1))^2|} \right)^b \right\} = 0$$

This implies that $f \notin G(a,b,\Phi(f))$, a contradiction. Hence, we must have $|w(z)| < 1$ for all $z \in D$. Therefore, any $f \in G(a,b,\Phi(f))$ is Φ-like univalent by (1). This completes the proof of the theorem.

Remarks: If Φ is the identity function and $a = \alpha$, $b = 1$, then we obtain the results in [3] and [4] due to Mocanu, Miller, and Reade. If $\Phi(f)$ is a starlike function defined in D then, by using Theorem 1, we obtain the subclass of close-to-convex functions in the sense of W. Kaplan [5].

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