Revista Colombiana de Matemáticas
Vol. XIII (1979) págs 1 - 38

HILBERT MODULES AND THEIR REPRESENTATION

by

Alonso TAKAHASHI

Abstract. A Hilbert module over a C^*-algebra A with identity is an unitary A-module H, endowed with an inner product taking values in A. We consider these modules and derive some of their elementary properties. Later we consider fields of Hilbert modules and obtain a representation of H by means of continuous sections. This result will be used in other papers ([7] and [8]) to get a duality theorem and to study special Hilbert modules.

§ 1. Introduction. Given a Hilbert space H and a topological space X let $C^b(X,H)$ be the set of all bounded continuous functions $s:X \to H$

and let $C^b(X)$ be the C^*-algebra of all bounded continuous complex functions on X. Then $C^b(X,H)$ is a $C^b(X)$-module with pointwise defined operations. In a similar way we can define an "inner product" on $C^b(X,H)$ with values in $C^b(X)$. Indeed, if $s,t \in C^b(X,H)$ then $\langle s|t\rangle : X \to C$ is defined by $\langle s|t\rangle(x) = \langle s(x)|t(x)\rangle$ for each $x \in X$. This inner product satisfies the formal properties of the inner product in an ordinary Hilbert space. We say that $C^b(X,H)$ is a "Hilbert module" and that $C^b(X)$ is its C^*-algebra of scalars.

Now, if $E = X \times H$ and $\pi : X \times H \to X$ is the projection onto the first factor we can identify $C^b(X,H)$ with the space $\Gamma^b(\pi)$ of all bounded continuous global sections in the fiber structure (E,π,X), i.e. the space of all bounded continuous maps $\sigma : X \to E$ with $\pi \circ \sigma = i_X$. Indeed, for each $s : X \to H$ in $C^b(X,H)$ we define $\sigma = \sigma_s : X \to E$ by letting $\sigma(x) = (x, s(x))$ for each $x \in X$. Then $\sigma \in \Gamma^b(\pi)$ and the map $s \mapsto \sigma_s$ is an isomorphism of $C^b(X,H)$ onto $\Gamma^b(\pi)$. Note that $\pi^{-1}(x) = \{x\} \times H \cong H$ for all $x \in X$, i,e. each fiber is (isomorphic to) the given Hilbert space H.

A fiber structure of the type just considered is called trivial. More generally we can consider fiber structures (E,π,X) such that each fiber $\pi^{-1}(x)$ is a Hilbert module over a C^*-algebra A_x, not necessarily the same for all $x \in X$. This kind of structure is called a field of Hilbert modules; it carries along a field (G, ρ, X) of C^*-algebras whose fibers or stalks are the A_x's. In this case $\Gamma^b(\pi)$ is a Hilbert module over $\Gamma^b(\rho)$, the C^*-algebra of all bounded continuous global sections of ρ. We prove that any Hilbert module over an arbitrary C^*-algebra A (with identity) can be represented as the Hilbert module of all bounded global sections in a canonical field π of Hilbert modules whose base space is the complete regularization of the primitive ideal space of A (which turns out to be homeomorphic to the maximal ideal space of the center of A). The essential tool we use is the general representation theory developed by Dauns and Hofmann in [1] and in particular the generalization of the Gelfand-Naimark theorem given there.

Hilbert modules were introduced by Kaplansky in [6] under the name of C^*-modules and the attention was focused in the case in which the C^*-algebra of scalars is a commutative AW^*-algebra. This particular kind of Hilbert module (AW^*-modules or Kaplansky-Hilbert modules) was considered in [3] and [6] in the study of algebras of Type I. More recently J.D.M. Wright [10] has studied operators on a Kaplansky-Hilbert module.

On the other hand, uniform fields of Hilbert spaces were used by Godement in [4] and were further studied by Dixmier and Douady in [3]. There the theory was also applied to the study of C^*-algebras.

The representation of Hilbert modules mentioned above is used to establish a close relationship between Hilbert modules and fields of Hilbert spaces. In fact, it can be proved ([7]) that with a suitable definition of morphism, the class $\mathcal{M}$ of all Hilbert modules with commutative C^*-algebra of scalars and the class $\mathcal{F}$ of all fields of Hilbert spaces with compact base space become categories, and these categories are equivalent.

These ideas can be used ([8]) to study the Kaplansky-Hilbert modules by considering fields of Hilbert spaces with compact extremally disconnected base space.

The ideas of this paper appeared first in the author's doctoral dissertarion, written at Tulane University in 1971 under Professor K.H. Hofmann to whom the author is grateful. Some of our results are stated in Professor Hofmann's survey "Representation of Algebras By Continuous Sections", Bull of the A.M.S. 78, 3 (1972) pp. 291-372.

§ 2. Hilbert modules. We first introduce positive hermitian forms with values in a C^*-algebra A, and prove a generalized version of the Cauchy-Schwarz inequality. From this result many of the subsequent properties follow using formally the same arguments as in the case of ordinary inner product spaces.

Let A be a C^*-algebra with identity and let H be a unitary module over A ; we say that A is the *C^*-algebra of scalars* of H. Since $i = i_A \in A$ we can, and will, identify each $\lambda \in \mathbb{C}$ with $\lambda \cdot 1_A \in A$ and so H becomes a complex vector space.

2.01. Definition. A *hermitian form on* H *with values in* A is a map $\langle|\rangle : H \times H \to A$ such that for all $x, x', y \in H$ and all $a \in A$, the following conditions hold:

(1) $\langle x+x'|y\rangle = \langle x|y\rangle + \langle x'|y\rangle$,

(2) $\langle ax|y\rangle = a\langle x|y\rangle$,

(3) $\langle y|x\rangle = \langle x|y\rangle^*$.

It easily follows that:

(1') $\langle x|y+y'\rangle = \langle x|y\rangle + \langle x|y'\rangle$.

(2') $\langle x|ay\rangle = \langle x|y\rangle a^*$,

(3') $\langle x|x\rangle$ is self adjoint, and

(2'') for all $\lambda \in \mathbb{C}$, $\langle \lambda x | y \rangle = \lambda \langle x | y \rangle$ and

$\langle x | \lambda y \rangle = \bar{\lambda} \langle x | y \rangle$.

A pair of elements $x, y \in H$ is said to be *orthogonal* (with respect to the hermitian form $\langle | \rangle$), if $\langle x | y \rangle = 0$. Given any subset $S \subseteq H$, the set of all $x \in H$ which are orthogonal to each one of the elements of S is denoted by $S^{\perp}$.

The hermitian form $\langle | \rangle$ is said to be *positive* if

(4) for all $x \in H$, $\langle x | x \rangle \in A^{+}$.

It is *definite* if

(5) for all $x \in H$, $\langle x | x \rangle = 0 \Longrightarrow x = 0$.

A positive definite hermitian form is called an *inner product* (with values in A).

2.02. Proposition. (The Cauchy-Schwarz inequality). *If* $\langle | \rangle$ *is a positive hermitian form on* H *then*

$$\langle x | y \rangle \langle x | y \rangle^{*} \leqslant \| \langle y | y \rangle \| . \langle x | x \rangle$$

for any pair of elements x,y *in* H.

Proof. Take $a \in A$ and $\varepsilon > 0$ arbitrary. We have

$$0 \leqslant \langle x - ay | x - ay \rangle = \langle x | x \rangle - \langle x | y \rangle a^{*} - a \langle y | x \rangle + a \langle y | y \rangle a^{*} .$$

Since $aa^* \geqslant 0$ we also have

$$0 \leqslant \langle x|x\rangle - \langle x|y\rangle a^* - a\langle x|y\rangle^* + a(\langle y|y\rangle + \varepsilon . 1_A)a^* .$$

Let $M = \sup[\mathrm{Sp}\langle y|y\rangle] = \|\langle y|y\rangle\| \geqslant 0$, so that $\sup[\mathrm{Sp}(\langle y|y\rangle + \varepsilon . 1_A)] = M + \varepsilon$.

If we define $z = \langle y|y\rangle + \varepsilon . 1_A$ then z^{-1} exists and $\inf[\mathrm{Sp}\ z^{-1}] = 1/(M+\varepsilon)$. Thus $[1/(M+\varepsilon)].1_A \leqslant z^{-1}$.

Now take $a = \langle x|y\rangle z^{-1}$. Since z is positive, in particular self-adjoint, we have $a^* = z^{-1}\langle x|y\rangle^*$. Replacing a and a^* in the inequality above we get:

$$0 \leqslant \langle x|x\rangle - \langle x|y\rangle z^{-1}\langle x|y\rangle^* - \langle x|y\rangle z^{-1}\langle x|y\rangle^*$$
$$+ \langle x|y\rangle z^{-1}\ z\ z^{-1}\ \langle x|y\rangle^* ,$$

that is $0 \leqslant \langle x|x\rangle - \langle x|y\rangle z^{-1}\langle x|y\rangle^*$.

Thus $\langle x|y\rangle z^{-1}\langle x|y\rangle^* \leqslant \langle x|x\rangle$. But in any C^*-algebra A we have $c\ a\ c^* \leqslant c\ b\ c^*$ whenever $a \leqslant b$, $a,b,c \in A$, ([2], 1.6.8). Then, from $0 < [1/(M+\varepsilon)]1_A \leqslant z^{-1}$ it follows that

$$0 \leqslant \langle x|y\rangle(\frac{1}{M+\varepsilon}.1_A)\ \langle x|y\rangle^* \leqslant \langle x|y\rangle z^{-1}\langle x|y\rangle^* .$$

Then $0 \leqslant \frac{1}{M+\varepsilon}\ \langle x|y\rangle\langle x|y\rangle^* \leqslant \langle x|x\rangle$ that is ,

$0 \leq \langle x|y\rangle \langle x|y\rangle^* \leq (M+\varepsilon) \langle x|y\rangle$.

Since A^+ is closed ([2], 1.6.1), we obtain

$$0 \leq \langle x|y\rangle \langle x|y\rangle^* \leq M \langle x|x\rangle . \blacksquare$$

When A is commutative we can get a sharper result:

2.3. Corollary. (The commutative case). *If in the above proposition* A *is abelian, then*

$$\langle x|y\rangle\langle x|y\rangle^* \leq \langle x|x\rangle \langle y|y\rangle .$$

Proof. An in the proof of the proposition we obtain

$$\langle x|y\rangle z^{-1} \langle x|y\rangle^* \leq \langle x|x\rangle .$$

Multiplying this inequality by z we get

$$\langle x|y\rangle\langle x|y\rangle^* \leq \langle x|x\rangle\langle y|y\rangle + \varepsilon \langle x|x\rangle ,$$

for arbitry $\varepsilon > 0$. This implies the corollary. ■

2.04. Example. Set $H = A =$ all 2×2 complex matrices, and for $x,y \in H$ define $\langle x|y\rangle = x y^*$. This is a positive hermitian form on H with values in $A = H$ and if we take

$$x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then $x, y \in H$, but

$$\langle x|x\rangle\langle y|y\rangle - \langle x|y\rangle\langle x|y\rangle^{*} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which is not positive. This example shows that the more pleasant result 2.03 is not necessarily true for A non-commutative.

2.05. Proposition. (Numerical Cauchy-Schwarz inequality). *If* $\langle\,|\,\rangle$ *a positive hermitian form on* H *and* $x, y \in H$ *then,*

$$\|\langle x|y\rangle\| < \|\langle x|x\rangle\|^{\frac{1}{2}} \, \|\langle y|y\rangle\|^{\frac{1}{2}}$$

Proof. Since in any C^{*}-algebra, $a, b \in A^{+}$ and $a \leqslant b$ imply $\|a\| \leqslant \|b\|$ ([2], 1.6.9), the assertion follows from 2.02. ∎

2.06. Lemma. *Let* $\langle\,|\,\rangle$ *be a positive hermitian form on* H *with values in* A, *let* m *be a closed 2-sided ideal of* A. *Then for each* $x \in H$, *the following assertions are equivalent :*

(i) $\langle x|x\rangle \in m$.

(ii) $\forall y \in H : \langle x|y\rangle \in m$.

(iii) $\forall y \in H : \langle y|x\rangle \in m$.

<u>Proof</u>. Clearly (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i). So let us prove (i) $\Rightarrow$ (ii): There exists a family $\{\pi_i\}_{i\in I}$ of non-null irreducible representations of A such that $m = \bigcap_{i\in I} \ker(\pi_i)$, ([2], 2.9.7). Since the π_i's are C^*-homomorphisms they preserve products, adjoint and positiveness, and so the Cauchy-Schwarz inequality $\langle x|y\rangle\langle x|y\rangle^* \leq \|\langle y|y\rangle\| \langle x|x\rangle$ implies:

$$0 \leq \pi_i(\langle x|y\rangle)\pi_i\langle x|y\rangle)^* = \pi_i(\langle x|y\rangle\langle x|y\rangle^*)$$

$$\leq \pi_i(\|\langle y|y\rangle\|\langle x|x\rangle) = \|\langle y|y\rangle\|\pi_i(\langle x|x\rangle).$$

So, if we suppose $\langle x|x\rangle \in m$, i.e. $\pi_i(\langle x|x\rangle) = 0$ for all $i \in I$, we have

$$\|\pi_i(\langle x|y\rangle)\| = \|\pi_i(\langle x|y\rangle)\ \pi_i(\langle x|y\rangle)^*\|^{\frac{1}{2}} = 0$$

that is $\pi_i(\langle x|y\rangle) = 0$, for all $i \in I$. Then $\langle x|y\rangle \in m$ (for arbitrary $y \in H$). ∎

2.07. <u>Lemma</u>. *In the setting of* 2.06 *define* $H_m = \{x \in H : \langle x|x\rangle \in m\}$. *Then* : (i) H_m *is an* A-*submodule of* H , *and* (ii) $mH \subseteq H_m$.

<u>Proof</u>: First suppose $x,y \in H_m$ and $a \in A$. We have $\langle x+y|x+y\rangle = \langle x|x\rangle+\langle x|y\rangle+\langle y|x\rangle+\langle y|y\rangle$ where $\langle x|y\rangle,\langle y|x\rangle \in m$ by 2.06. Thus $\langle x+y|x+y\rangle \in m$, that is $x+y \in H_m$. Also $\langle ax|ax\rangle = a\langle x|x\rangle a^* \in m$ because $\langle x|x\rangle \in m$ and this is an ideal. Now take $x \in H$ and $a \in m$. Then $\langle ax|ax\rangle = a\langle x|ax\rangle \in m$.

2.08. <u>Proposition</u>. *Let* $\langle|\rangle$ *be a positive hermitian form on* H *with values in* A *and let* m *be a closed 2-sided ideal of* A . *Take* $\xi,\eta \in H/H_m$ *and* $\alpha \in A/m$, *say* $\xi = x+H_m$, $\eta = y+H_m$ *and* $\alpha = a+m$ *where* $x,y \in H$ *and* $a \in A$. *Define:*

$$\xi + \eta = x+y+H_m \qquad \alpha\xi = ax+H_m$$

$$(\xi|\eta) = \langle x|y\rangle + m .$$

Then these definitions are independent of the representatives; H/H_m *is a unitary* A/m-*module and* $(|)$ *is an inner product on* H/H_m *with values in* A/m.

<u>Proof</u>. Addition is well defined because H_m is an additive subgroup of H.
Now suppose $a-a' \in m$ and $x-x' \in H_m$, then :

$$ax - a'x' = a(x-x')+(a-a')x' .$$

But $a(x-x') \in H_m$ because this is an A-submodule. Also $(a-a')x' \in H_m$ because $mH \subseteq H_m$ (2.07). Thus $ax-a'x' \in H_m$. For the third case take $x-x'$ and $y-y'$ in H_m. We have

$$\langle x|y\rangle - \langle x'|y'\rangle = \langle x-x'|y\rangle + \langle x'|y-y'\rangle$$

and since $x-x' \in H_m \Leftrightarrow \langle x-x'|x-x'\rangle \in m$ then $\langle x-x'|y\rangle \in m$, by 2.06. Similarly $\langle x'|y-y'\rangle \in m$, and so $\langle x|y\rangle - \langle x'|y'\rangle \in m$. We readily check that H/H_m is then a unitary A/m-module and that $(|)$ is a positive hermitian form. It is also definite, indeed, for $\xi = x+H_m$ we have

$$(\xi|\xi) = 0 \ (=m) \Leftrightarrow \langle x|x\rangle \in m$$

$$\Leftrightarrow x \in H_m$$

$$\Leftrightarrow \xi = 0 \ (=H_m).$$

This completes the proof. ∎

Since there is no danger of confusion we will write $\langle|\rangle$ instead of $(|)$. The proposition just proved will be needed later; now we will use a particular instance of it to describe the canonical way to get an inner product out of a positive hermitian form.

2.09. Corollary. *Let $<|>$ be a positive hermitian form on* H *with values in* A *and let* $N = \{x \in H: <x|x> = 0\}$. *Then* N *is an* A*-submodule of* H, *and if for* $x+N$ *and* $y+N$ *in* H/N *and for* $a \in A$ *we define*

$$(x+N)+(y+N)=x+y+N, \quad a(x+N)=ax+N, \quad (x+N|y+N)=<x|y>,$$

then H/N *is a unitary* A*-module and* $(|)$ *is an inner product on* H/N *with values in* A.

Proof. It suffices to appy 2.07 and 2.08 with $m = \{0\}$. ∎

Let H be a unitary A-module and $<|>$ be an inner product on H with values in A. We will discuss the following two functions:

a) The A-valued "norm" :

$$| \; | : H \to A^+$$

given by $|x| = <x|x>^{\frac{1}{2}}$ for each $x \in H$, and

b) The numerical norm:

$$\| \; \| : H \to \mathbb{R}^+$$

given by $\|x\| = \|<x|x>\|^{\frac{1}{2}}$ $(= \| |x| \|)$ for each $x \in H$.

2.10. Proposition. (Norm properties of $| \; |$ and

$\| \; \|$). *Take* x, y *in* H *and* a *in* A.
Then:

1) (i) $|x| = 0 \Leftrightarrow x = 0$

if A *is commutative:*

(ii) $|x+y| \leqslant |x| + |y|$, *and*

(iii) $|ax| = |a| \; |x|$, *where* $|a| = (a\, a^*)^{\frac{1}{2}}$.

2) (i) $\|ax\| \leqslant \|a\| \; \|x\|$, *and*

(ii) $\|\langle x|y\rangle\| \leqslant \|x\| \; \|y\|$.

3) *The function* $\| \; \|$ *is an ordinary norm on* H.

Proof. 1) (i) $|x| = \langle x \; x\rangle^{\frac{1}{2}} = 0 \Leftrightarrow \langle x|x\rangle = 0 \Leftrightarrow x = 0$.

(ii) Taking 2.03 into account we have:

$$\begin{aligned}\langle x+y|x+y\rangle &= \langle x|x\rangle + \langle x|y\rangle + \langle x|y\rangle^* + \langle y|y\rangle \\ &\leqslant \langle x|x\rangle + 2(\langle x|y\rangle\langle x|y\rangle^*) + \langle y|y\rangle \\ &\leqslant \langle x|x\rangle + 2\langle x|x\rangle^{\frac{1}{2}}\langle y|y\rangle^{\frac{1}{2}} + \langle y|y\rangle \\ &= (\langle x|x\rangle^{\frac{1}{2}} + \langle y|y\rangle^{\frac{1}{2}})^2 .\end{aligned}$$

(iii) This follows from the equation $\langle ax|ax\rangle = a\, a^* \langle x|x\rangle$.

2) (i) $\|ax\|^2 = \|\langle ax|ax\rangle\| = \|a\langle x|x\rangle a^*\|$

$\leqslant \|a\| \; \|\langle x|x\rangle\| \; \|a^*\| = \|a\|^2 \; \|x\|^2$.

(ii) This is simply 2.05.

3) Clearly $\|x\| = 0$ iff $x = 0$, and $\|\lambda x\| = |\lambda|\,\|x\|$ for $\lambda \in \mathbb{C}$. As for the triangle inequality:

$$\begin{aligned} \|x+y\|^2 &= \|\langle x+y|x+y\rangle\| \\ &\leqslant \|\langle x|x\rangle\| + \|\langle x|y\rangle\| + \|\langle x|y\rangle^*\| + \|\langle y|y\rangle\| \\ &< \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{, by 2) (ii),} \\ &= (\|x\|+\|y\|)^2. \blacksquare \end{aligned}$$

2.11. <u>Corollary</u>. *The map* $A \times H \to H$, $(a,x) \mapsto ax$ *is continuos*.

<u>Proof</u>. This follows from 2) (i). ■

Since we will use it several times later, we state the following simple inequatily as a lemma:

2.12. <u>Lemma</u>. *For any* $x, y, s, t \in H$:

$$\|\langle x|x\rangle - \langle s|t\rangle\| \leqslant \|x-s\|\|y\| + \|s\|\|y-t\| .$$

<u>Proof</u>. Since $\langle x|y\rangle - \langle s|t\rangle = \langle x-s|y\rangle + \langle s|y-t\rangle$, the inequality follows from the triangle inequality and 2.10, 2) (ii). ■

2.13. <u>Proposition</u>. *The inner product* $H \times H \to A$, $(x,y) \mapsto \langle x|y\rangle$ *is continuos*.

<u>Proof</u>. Fix $s, t \in H$ and $0 < \varepsilon \leqslant 1$. If we take

$x,y \in H$ with $\|x-s\| < \varepsilon$ and $\|y-t\| < \varepsilon$, the lemma above implies

$$\|\langle x|y\rangle - \langle s|t\rangle\| \leq \varepsilon(1+\|s\|+\|t\|). \blacksquare$$

2.14. Corollary. *The map* $H \to A$, $x \mapsto \langle x|x\rangle$ *is continuos.*

Proof. The map $H \to H \times H$, $x \mapsto (x,x)$ is continuous.$\blacksquare$

The $\|\cdot\|$ defined in the last section allows us to discuss completeness of H, introducing in this way the notion of Hilbert module. We will also consider the process of completing a given module endowed with an inner product.

As before A will be a C^*-algebra with identity.

2.15. Definition. A *pre-Hilbert module over* A is a unitary A-module H together with a positive definite hermitian form $\langle|\rangle$ with values in A. It is called a *Hilbert module* if it is complete with respect to the norm $\|x\| = \|\langle x|x\rangle\|^{\frac{1}{2}}$, $x \in H$.

Since a pre-Hilbert module H is in particular a normed space we can construct its completion $\tilde{H}$ in the usual way by throwing in all "limits" of Cauchy sequences in H, so that H will be a dense sub-space of $\tilde{H}$. In order to extend the inner product to $\tilde{H}$ take x,y in $\tilde{H}$ and choose $\{x_n\}$, $\{y_n\} \subseteq H$ with $\|x-x_n\| \to 0$ and $\|y-y_n\| \to 0$. Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and in particular they are bounded, say $\|x_n\|, \|y_n\| \leq M$ for all n.

Taking $x = x_n$, $y = y_n$, $s = x_m$ and $t = y_m$ in 2.12 we have $\|<x_n|y_n>-<x_m|y_m>\| < M(\|x_n-x_m\| + \|y_n-y_m\|)$ which proves that $\{<x_n|y_n>\}$ is a Cauchy sequence in A. Since A is complete we can define

$$<<x|y>> = \lim_{m\to\infty} <x_m|y_m> ,$$

provided that this limit is independent of the particular sequences $\{x_n\}, \{y_n\}$. To see this take $x_n' \to x$ and $y_n' \to y$, so that $\|x_n-x_n'\| \to 0$ and $\|y_n-y_n'\| \to 0$. Another application of 2.12 yields

$$0 \leq \|<x_n|y_n>-<x_n'|y_n'>\| \leq \|x_n-x_n'\|\|y_n\| + \|x_n'\|\|y_n-y_n'\| \to 0$$

i.e. $\lim_{n\to\infty} <x_n'|y_n'> = \lim_{n\to\infty} <x_n|y_n>$.

We readily check that $<<|>>$ is in fact an inner product on $\tilde{H}$. The norm $|||\ |||$ induced by this inner product coincides with the norm $\|\ \|$ of $\tilde{H}$ considered as the completion of the normed space H.

Indeed, take $x \in \tilde{H}$ and $\{x_n\} \subseteq H$ with $\lim_{n\to\infty} x_n = x$, then

$$|||x|||^2 = \|<<x|x>>\| = \|\lim_{n\to\infty} <x_n|x_n>\| =$$

$$\lim_{n\to\infty} \|<x_n|x_n>\| = \lim_{n\to\infty} \|x_n\|^2$$

$$= (\lim_{n\to\infty} \|x_n\|)^2 = \|x\|^2.$$

We will write $<|>$ instead of $<<|>>$. Now we can state the facts discussed above in the following way:

2.16. <u>Proposition</u>. *The completion of a pre-Hilbert module can be made into a Hilbert module in a natural way.* ∎

2.17. <u>Examples</u>. (A) Take $H = A$ and define $<x|y> = xy^*$; $x,y \in A$. Then H is a Hilbert module over A.

(B) Let Z be the center of A and let τ be a central trace, i.e. a map $A \to Z$ such that (1) $\tau(\lambda x+y) = \lambda\tau(x)+\tau(y)$; (2) $\tau(zx) = z\tau(x)$; (3) $\tau(x^* x) \geqslant 0$ and $\tau(x^* x) = 0$ only if $x = 0$; (4) $\tau(x\ y) = \tau(y\ x)$, and (5) $\tau(1) = 1$, for all λ in $\mathbb{C}$, $x,y \in A$ and $z \in Z$.

Define

$$<x|y> = \tau(x\ y^*).$$

Then A is a pre-Hilbert module over Z.

(C) (Kaplansky) Let B be an AW^*-algebra of type I. Then there exists an abelian projection $e \in B$ with $ex \neq 0$ for all non-zero central elements of B. Let $H = eB$ and $A = eBe$. Then H is an A-module and if we set $\langle x|y\rangle = xy^*$ for all $x,y \in H$, it becomes a Hilbert module ([6], Theorem 8).

§ 3. Fields of Hilbert Modules. In this paragraph we discuss the structure of the set of all bounded continuous global sections in a field of Hilbert modules and after that we prove the representation theorem mentioned in the introduction. Although our definitions are essentially those of [1] we will state some of them in order to establish terminology in the particular situation we are interested in.

3.01 Let $\pi : E \to X$ be a surjective function. For each subset $V \subseteq X$ let $E_V = \pi^{-1}(V)$; if $V = \{x\}$ we write E_x instead of $E_{\{x\}} (= \pi^{-1}(x))$ and call it the *fiber* or stalk over x. A *section* (of π) *over* $V \subseteq X$ is any map $\sigma : V \to E$ such that $\pi \circ \sigma = 1_V$; if $V = X$ we say that σ is a *global section* (or simply a section) of π. The set of all section of π is denoted $\Sigma(\pi)$.

Now suppose that the E_x's are normed spaces. Since these fibers are pairwise disjoint no confusion arises from using the same symbol ‖ ‖ to denote the norm in all instances. For each $\sigma \in \Sigma(\pi)$ we define

$$\|\sigma\| = \sup_{x \in X} \|\sigma(x)\|$$

and we say that σ is *bounded* if $\|\sigma\| < +\infty$. The set $\Sigma^b(\pi) = \{\sigma \in \Sigma(\pi) : \sigma \text{ is bounded}\}$ is a normed space with respect to pointwise addition and scalar multiplication and the norm $\sigma \mapsto \|\sigma\|$. For each $\sigma \in \Sigma(\pi)$ and each $\varepsilon > 0$ the set

$$\mathcal{T}_{\varepsilon}(\sigma) = \{e \in E : \|e - \sigma[\pi(e)]\| < \varepsilon\}$$

is called the *ε-tube around* σ.

If E and X are topological spaces and the surjection $\pi : E \to X$ is continuos we say that the triple (E,π,X) is a *fiber structure*. We define $\Gamma(\pi) = \{\sigma \in \Sigma(\pi) : \sigma \text{ is continuos}\}$ and $\Gamma^b(\pi) = \{\sigma \in \Sigma(\pi) : \sigma \text{ is bounded and continuos}\}$ $= \Gamma(\pi) \cap \Sigma^b(\pi)$. The set $E \vee E = \{(e,e') : e, e' \in E, \pi(e) = \pi(e')\}$ is always considered as a subspace of $E \times E$.

We introduce a *field addition*: $E \vee E \to E$, $(e,e') \mapsto e + e'$, and a *field scalar multiplication*: $C \times E \to E$, $(\lambda,e) \mapsto \lambda e$.

The fiber structure (E,π,X) is called a *field of normed spaces* (cf. [1] Ch. I, §1 and Ch. III, §8) if the following conditions are satisfied:

(F.1) For each $e \in E$ there exists a $\sigma \in \Gamma^b(\pi)$ with $\sigma[\pi(e)] = e$ (we say that $\Gamma^b(\pi)$ is *full*) and for such a σ the sets of the form

$$\mathcal{T}_\varepsilon(\sigma) \cap E_V$$

where $\varepsilon > 0$ and V is a neighborhood of $\pi(e)$ in X, constitute a basis for the neighborhoods of e in E.

(F.2) The field addition and scalar multiplication are continuos.

The following facts follow from the definitions:

(i) The set $\Gamma^b(\pi)$ is a normed space (subpace of the normed space $\Sigma^b(\pi)$). It is also a topological $C^b(X)$-module.

(ii) If all the stalks $E_x, x \in X$, are complete so are $\Gamma(\pi)$ and $\Gamma^b(\pi)$. In fact $\Gamma^b(\pi)$ is closed in $\Gamma(\pi)$ ([1] Ch. II, 1.13).

(iii) If X is compact then $\Gamma^b(\pi) = \Gamma(\pi)$ (ibid.).

(iv) Let X' be the same set X with a finer topology and let E' be the set E with the topology generated by the sets of the form $W \cap E_{V'}$, W open in E, V' open in X'. Then $\pi' = \pi: E' \to X'$

is also a field of normed spaces and $\Gamma(\pi)$ is a closed subspace of $\Gamma(\pi')$, ([1] Ch. I, 1.17).

<u>Proof of</u> (i) $a \in C^b(X)$ and $\sigma, \tau \in \Gamma^b(\pi)$ then $a\sigma$ and $\sigma + \tau$ are clearly in $\Sigma^b(\pi)$. Since the maps $a \times \sigma: X \to C \times E$ with $x \mapsto (a(x), \sigma(x))$ and $\sigma \times \tau: X \to E \times E$ with $x \mapsto (\sigma(x), \tau(x))$ are continuous, condition (2) implies the continuity of $a\sigma$ and $\sigma+\tau$. Thus $a\sigma$ and $\sigma+\tau$ are in $\Gamma^b(\pi)$ and so this is a $C^b(X)$-module. In particular $\lambda\sigma \in \Gamma^b(\pi)$ for any $\lambda \in C$ (take $a(x) = \lambda$ for all $x \in X$) so that $\Gamma^b(\pi)$ is a subspace of $\Sigma^b(\pi)$.

Finally, since $\|a(x)\sigma(x)\| = \|a(x)\| \|\sigma(x)\| \leqslant \|a\| \|\sigma\|$, we see that $\|a\sigma\| \leqslant \|a\| \|\sigma\|$. Thus the map

$$C^b(X) \times \Gamma^b(\pi) \to \Gamma^b(\pi), \quad (a, \sigma) \mapsto a\sigma$$

is continuous. ∎

In order to impose more structure on our field (E, π, X) let each fiber E_x be a C^*-algebra. Now we can define a *field multiplication:*

$$E \vee E \to E \quad , \quad (e, e') \mapsto e\, e'$$

and a *field involution*

$$E \to E \quad , \quad e \mapsto e^* \ .$$

We make a third assumption:

(F.3) Field multiplication and involution are continuous.

If for $\sigma, \tau \in \Sigma(\pi)$ we define σ^* and $\sigma\tau$ pointwise then we can prove.

(v) The space $\Gamma^b(\pi)$ is a C^*-algebra.

Now we want to describe the non-commutative generalization of the Gelfand-Naimark theorem obtained by Dauns and Hofmann in [1] (8.13 and 8.14). Let A be a C^*-algebra with identity and let X be the maximal ideal space of the center Z of A. This is a (Hausdorff) compact space and it is, up to homeomorphism the complete regularization and the Hausdorffization of Prim A, the primitive ideal space of A. For each $x \in X$ (note that $x \subseteq Z \subseteq A$) let $m(x)$ be the closed 2-sided ideal generated by x and let $M = \{m(x) : x \in X\}$; the map $x \mapsto m(x)$ is bijective: in fact $x = m(x) \cap Z$ for each $x \in X$. We will suppose that the family $A/m(x)$, $x \in X$, of C^*-algebras is pairwise disjoint (if not we will consider $\{x\} \times A/m(x)$ instead).

Let $E = \bigcup_{x \in X} A/m(x)$ and $\pi : E \to X$ be given by $\pi(e) = x$ if $e \in A/m(x)$.

For each $a \in A$ we define a map

$$\check{a} : X \to E \quad , \quad x \mapsto a+m(x)$$

Clearly $\check{a} \in \Sigma(\pi)$ for all $a \in A$. The sets of the form $\mathcal{T}_\varepsilon(\check{a}) \cap E_V$ where $\varepsilon > 0$, $a \in A$ and V is an open subset of X constitute a basis for a topology on E and then (E,π,X) becomes a field of C^*-algebra with fibers $E_x = A/m(x)$, $x \in X$. In this field all sections $\check{a}$, $a \in A$, are continuous and the map

$$\check{} : A \to \Gamma(\pi), \quad a \mapsto \check{a}$$

is a C^*-algebra isomorphism. Then we can identify A with $\check{A} = \{\check{a} : a \in A\}$ $(=\Gamma(\pi))$. Under this isomorphism Z is mapped onto $C(X)$. $\check{1} \cong C(X)$.

We note that for each $\alpha \in \Gamma(\pi)$ the function $X \to \mathbb{R}$, $x \mapsto \|\alpha(x)\|$ is upper semi-continuous ([1] Ch. III, 5.9). This fact is useful to prove the following technical assertion:

3.02. <u>Lemma</u>. *For each* $a \in A$ *and each* $x \in X$

$$\|a+m(x)\| = \inf_{t \in x} \|(1+t)^2 a\|.$$

<u>Proof</u>. Given $t \in x$ we have $(1+t)^2 a - a = t(2a+ta) \in x A \subseteq m(x) A = m(x)$, i.e. $(1+t)^2 a$ is a representative of the coset $a+m(x)$. Hence $\|a+m(x)\| \leq \|(1+t)^2 a\|$ (for arbitrary $t \in x$), that is $\|a+m(x)\| \leq \inf_{t \in x} \|(1+t)^2 a\|$.

To obtain the opposite inequality we identify

A with $\Gamma(\pi)$ via the isomorphism $\check{}$. Then x (resp. m(x)) is identified with $I(x) = \{\varphi \in C(X): \varphi(x)=0\}$ (resp. $J(x) = \{\alpha \in \Gamma(\pi):\alpha(x) = 0\}$). Noting that $a+m(x) = \check{a}(x)$, what we want is

$$\|\alpha(x)\| \geq \inf_{\varphi \in I(x)} \|(1+\varphi)^2\alpha\|, \text{ for } \alpha \in \Gamma(\pi), x \in X.$$

Fix $\varepsilon > 0$ and let $V = \{y \in X:\|\alpha(y)\|<\|\alpha(x)\|+\varepsilon\}$. Then V is open and contains x and so there exists a $\psi \in C(X)$ with $0 \leq \psi \leq 1$, $\psi(x) = 1$ and $\operatorname{supp} \psi \subseteq V$.

Take $\varphi = \sqrt{\psi}-1$; then $\varphi(x) = \sqrt{\psi(x)}-1 = 0$, i.e. $\varphi \in I(x)$. Moreover:

$$\|(1+\varphi(y))^2\alpha(y)\| = \|\psi(y)\alpha(y)\| = |\psi(y)|\,\|\alpha(y)\| \leq \|\alpha(x)\|+\varepsilon$$

if $y \in V$, and is 0 if $y \notin V$.

So $\|(1+\varphi(y))^2\alpha(y)\|<\|\alpha(x)\|+\varepsilon$ for all $y \in X$, that is $\|(1+\varphi)^2\alpha\| \leq \|\alpha(x)\|+\varepsilon$. Then $\inf_{\varphi \in I(x)} \|(1+\varphi)^2\alpha\| \leq \|\alpha(x)\|+\varepsilon$ and since ε was arbitrary we conclude that $\inf_{\varphi \in I(x)} \|(1+\varphi)^2\sigma\| \leq \|\alpha(x)\|$. ∎

3.03. Let (E,π,X) (or simply π) be a field of C^*-algebras with identity and let (F,ρ,X) $(=\rho)$ be a field of normed spaces in which each fiber $F_x = \rho^{-1}(x)$ is actually a Hilbert module over the C^*-algebra $E_x = \pi^{-1}(x)$. The inner product on each fiber is denoted by one and the same symbol $<|>$.

The *field scalar multiplication* can be defined in the subspace

$E \vee F = \{(e,f) \in E \times F : \pi(e) = \rho(f)\}$ of $E \times F$ as follows:

$$E \vee F \to F \quad , \quad (e,f) \mapsto ef \; .$$

We also define a *field inner product* $<|>$ by

$$F \vee F \to E \quad , \quad (f,f') \mapsto <f|f'>$$

3,04, Definition. If the field scalar multiplication and inner product are continuous the pair (π,ρ) is said to be a *field of Hilbert modules*. When each stalk F_x is a Hilbert space, i.e. $E_x = \mathbb{C}$ for all $x \in X$, we say that (π,ρ) is a *field of Hilbert spaces*.

As in 3.01 (i) we can prove that the Banach space $\Gamma^b(\rho)$ (see 3.01(ii)) is a topological $\Gamma^b(\pi)$-module.

Given $\sigma,\tau \in \Gamma^b(\rho)$ the map $X \to F \vee F$, $x \mapsto (\sigma(x),\tau(x))$ is continuous and so the continuity of the field inner product implies the continuity of the map $[\sigma|\tau] : X \to E$ given by $[\sigma|\tau](x) = <\sigma(x)|\tau(x)>$. This map is clearly a section of π and by the numerical Cauchy-Schwarz inequality

$$\|[\sigma|\tau](x)\| = \|<\sigma(x)|\tau(x)>\| \leqslant \|\sigma(x)\|\,\|\tau(x)\| \leqslant \|\sigma\|\,\|\tau\| < +\infty ,$$

hence $[\sigma|\tau] \in \Gamma^b(\pi)$. Thus we have a map $[\ |\ ]$: $\Gamma^b(\rho) \times \Gamma^b(\rho) \to \Gamma^b(\pi)$ and we readily check that this is an inner product on $\Gamma^b(\pi)$.

3.05. Proposition. *The space* $\Gamma^b(\rho)$ *is a Hilbert module over* $\Gamma^b(\pi)$.

Proof. We only need to show that $\Gamma^b(\rho)$ is complete with respect to the norm $\|\ \|_1$ induced by $[|]$. Since it is complete with respect to the norm $\|\sigma\| = \sup_{x\in X}\|\sigma(x)\|$ (see 3.01 (ii)), it is enough to show that these norms coincide. Take $\sigma \in \Gamma^b(\pi)$:

$$\|\sigma\|_1^2 = \|[\sigma|\sigma]\| = \sup_{x\in X}\|[\sigma|\sigma](x)\| = \sup_{x\in X}\|<\sigma|(x)|\sigma(x)>\|$$

$$= \sup_{x\in X}\|\sigma(x)\|^2 = (\sup_{x\in X}\|\sigma(x)\|)^2 = \|\sigma\|^2. \blacksquare$$

We will write $<|>$ instead of $[|]$.

3.06. Definition. The $\Gamma^b(\pi)$-Hilbert module $\Gamma^b(\rho)$ is called the *Hilbert module associated* with the field of Hilbert modules (π,ρ).

In this sections we will associate a canonical field of Hilbert modules to each A-Hilbert module H. The C^*-algebra A is supposed to have an identity and the notation is that of 3.01; in order to avoid confusion between elements of X and elements of H the latter will be denoted by $u, v, \ldots$ etc.

Fix $x \in X$. We will write H_x instead of $H_{m(x)}$ (see 2.07), in other words $H_x = \{u \in H : \langle u|u\rangle \in m(x)\}$. Since $H \to A$, $u \mapsto \langle u|u\rangle$ is continuous (2.14) and $m(x)$ is closed, H_x is closed. Then the quotient space H/H_x is a Banach space with respect to the usual quotient norm; denote this norm by $\| \ \|_q$.

On the other hand, in 2.08 we defined an inner product on H/H_x with values in $A/m(x)$ and this inner product induces a norm on H/H_x which we will call $\|\cdot\|_i$.

3.07. <u>Lemma</u>. *The norms* $\| \ \|_q$ *and* $\| \ \|_i$ *coincide.*

<u>Proof</u>. Take $\xi \in H/H_x$ arbitrary and let $u \in \xi$, i.e. $u + H_x = \xi$.

a) $\|\xi\|_i^2 = \|\langle\xi|\xi\rangle\| = \|\langle u|u\rangle + m(x)\| = \inf\{\|\langle u|u\rangle + t\| :$

$t \in m(x)\} \leq \|\langle u|u\rangle\| = \|u\|^2$,

thus $\|\xi\|_i \leq \|u\|$, for any $u \in \xi$. It follows that $\|\xi\|_i \leq \inf\{\|u\| : u \in \xi\} = \|\xi\|_q$.

b) Now take $\varepsilon > 0$ arbitrarily and let $a = \langle u|u\rangle$ $(\in A)$; then $\|\xi\|_i^2 = \|a + m(x)\|$ and we can use 3.02 to produce an element $t \in x$ such that $\|(1+t)^2 a\| < \|\xi\|_i^2 + \varepsilon$. Define $v_o = (1+t)u$, so that $v_o - u = tu \in xH \subseteq m(x)H \subsetneq H_x$ (by 2.07 (ii)); thus $v_o \in \xi$.

Furthermore:

$$\|v_o\|^2 = \|\langle v_o | v_o\rangle\| = \|(1+t)\langle u|u\rangle(1+t)\| = \|(1+t)^2 a\| < \|\xi\|_i^2 + \varepsilon.$$

Hence $\inf\{\|v\|^2 : v \in \xi\} \leqslant \|\xi\|_i^2 + \varepsilon$ and, since $\varepsilon > 0$ was arbitrary, $\inf\{\|v\|^2 : v \in \xi\} \leqslant \|\xi\|_i^2$. Then $\|\xi\|_q = \inf\{\|v\| : v \ \xi\} \leqslant \|\xi\|_i$. ∎

Since there is no danger of confusion we will write $\| \ \|$ for $\| \ \|_q (= \| \ \|_i)$.

3.08. Proposition. *For any* $x \in X$ *the quotient* H/H_x *is an* $A/m(x)$*-Hilbert module.*

Proof. The only detail to check is completeness of H/H_x, but this follows from the lemma and the fact that this quotient is a Banach space. ∎

Let us proceed to define a field whose stalks are precisely the Hilbert modules H/H_x; we suppose these are pairwise disjoint. If not we could replace H/H_x by $\{x\} \times H/H_x$.

Let $F = \bigcup_{x \in X} H/H_x$ and $\rho : F \to X$ be the surjection given by $\rho_o(f) = x$ whenever $f \in H/H_x$. For each $u \in H$ the map

$$\hat{u} : X \to F \quad , \quad x \mapsto u + H_x$$

is a section of ρ_o. Let $\hat{H} = \{\hat{u}: u \in H\}$.

Now we will apply the machinery developed by Dauns and Hofmann in [1] to construct the so called *canonical field* associated with H and the family $\{H_x: x \in X\}$. According to [1] Ch. II, 1.15 there are unique weakest topologies $N(\hat{H})$ and $T(\hat{H})$ on F and X respectively making $\rho_o: F \to X$ into a field of normed spaces with $\hat{H} \subseteq \Gamma^b(\rho_o)$.

A subbasis for $T(\hat{H})$ is given by the sets of the form

$$\{x \in X : \|\hat{u}(x)\| < \varepsilon\} , \quad u \in H , \quad \varepsilon > 0 .$$

A basis for $N(\hat{H})$ is given by the sets

$$\mathcal{T}_\varepsilon(u) \cap F_V , \quad u \in H , \quad \varepsilon > 0 , \quad V \in T(\hat{H}).$$

3.09. <u>Lemma</u>. *The topology* $T(\hat{H})$ *on* X *is contained in the original topology of* X.

<u>Proof</u>. Fix $u \in H$ and $\varepsilon > 0$ and let $a = \langle u|u\rangle$ $(\in A)$. Since $\|\hat{u}(x)\| = \|u + H_x\| = \|\langle u|u\rangle + m(x)\|^{\frac{1}{2}} = \|a + m(x)\| = \|\check{a}(x)\|$ then $\{x \in X: \|\hat{u}(x)\| < \varepsilon\} =$

$= \{x \in X : \|\check{a}(x)\| < \varepsilon\}$ is open in X because $\check{a}$ is upper semicontinuous. This shows that any subbasic set of $T(\hat{H})$ is open in the original topology of X and the lemma is proved. ∎

This lemma allows us to apply the process mentioned in 3.01 (iv): We refine the topology $T(\hat{H})$ up to the original topology of X and take as new open sets in F those of the form $W \cap F_V$ where $W \in N(\hat{H})$ and V is open in X. Then a basis for the new topology on F is given by the sets

$$\mathcal{T}_\varepsilon(u) \cap F_V \,, \quad u \in H \,, \quad \varepsilon > 0, \quad V \text{ open in } X.$$

Let ρ be the map ρ_o when we consider F with the new topology (and X with its original topology). Then $\rho : F \to X$ is a field of normed spaces and $\Gamma^b(\rho_o)$ is a closed subspaces of $\Gamma^b(\rho) = \Gamma(\rho)$. Thus we also have $\hat{H} \subseteq \Gamma(\rho)$.

Given an A-Hilbert module H we have constructed two fields :

(i) The field of C^*-algebras $\pi = (E, \pi, X)$ described in 3.01. Its fibers are $E_x = A/m(x)$, $x \in X$.

(ii) The field of normed spaces $\rho = (F, \rho, X)$ described above. For each $x \in X$ the fiber $F_x = H/H_x$ is a Hilbert module over E_x.

3.10. Proposition. *The pair (π,ρ) is a field of Hilbert modules.*

Proof. According to Definition 3.04 we must show that the field scalar multiplication and inner product are continuous.

(a) Fix $(e_o,f_o) \in E \vee F$ and let $x_o = \pi(e_o) = \rho(f_o) = \rho(e_o f_o)$. Take $a \in A$ (resp. $u \in H$) such that $\check{a}(x_o) = a+m(x_o) = e_o$ (resp. $\hat{u}(x) = u+H_{x_o} = f_o$), and set $v = au$; then $\hat{v}(x_o) = au+H_{x_o} = a+m(x_o))\cdot(u+H_{x_o}) = e_o f_o$. Thus a basis for the neighborhoods of $e_o f_o \in F$ is given by all the sets $\mathcal{T}_\varepsilon(\hat{v}) \cap F_V$, where $\varepsilon > 0$ and V is a neighborhood of x in X. Now fix ε and V.

For $(e,f) \in E \vee F$ let $x = \pi(e) = \rho(ef)$ and suppose that $\varepsilon \in \mathcal{T}_\delta(a) \cap E_U$ and $f \in \mathcal{T}_\delta(\hat{u}) \cap F_U$. where $\delta > 0$ and U is neighborhood of x contained in V. Then:

$\|ef-\hat{v}[\rho(ef)]\| = \|ef-\hat{v}(x)\| = \|ef-\check{a}(x)\ \hat{u}(x)\| \leq \|e-\check{a}(x)\| (\|f-\hat{u}(x)\|+\|\hat{u}(x)\|)+\|\check{a}(x)\|\ \|f-\hat{u}(x)\| \leq \delta(\delta+\|\hat{u}(x)\| +\|\check{a}(x)\|)$, and this will be $< \varepsilon$ if we choose δ and U carefully enough (for example, let $U_1 = \{x \in X : \|\hat{u}(x)\| < \|\hat{u}(x_o)\| + 1\}$, $U_2 = \{x \in X : \|\check{a}(x)\| < \|\check{a}(x_o)\| + 1\}$, $U = U_1 \cap U_2 \cap V$ and take $0 < \delta \leq 1$ such that $\delta < \varepsilon\ (\|\hat{u}(x_o)\|+\|a(x_o)\|+3)^{-1}$).

Then $e \in \mathcal{T}_\delta(\check{a}) \cap E_U$, $f \in \mathcal{T}_\delta(u) \cap F_U$ imply

$\| ef-\hat{v}[\rho(ef)] \| < \varepsilon$ and $x = \rho(ef) \in V$, that is $ef \in \mathcal{T}_{\varepsilon}(\hat{v}) \cap F_V$. q.e.d.

(b) Fix $(f_o, g_o) \in F \vee F$ and let $x_o = \rho(f_o) = \rho(g_o) = \pi(<f_o|g_o>)$. Take $u, v \in H$ with $\hat{u}(x_o) = u + H_{x_o} = f_o$ and $\hat{v}(x_o) = v + H_{x_o} = g_o$, and set $a = <u|v>$; then $\check{a}(x_o) = a + m(x_o) = <u|v> + m(x_o) = <u + H_{x_o} | v + H_{x_o}> = <f_o|g_o>$. Thus a basis for the neighborhoods of $<f_o|g_o> \in E$ is given by all the sets $\mathcal{T}_{\varepsilon}(\check{a}) \cap E_V$ where $\varepsilon > 0$ and V is a neighborhood of x_o in X. Now fix ε and V.

For $(f,g) \in F \vee F$ let $x = \rho(f) = \rho(g) = \pi(<f|g>)$ and suppose that $f \in \mathcal{T}_{\delta}(\hat{u}) \cap F_U$ and $g \in \mathcal{T}_{\delta}(v) \cap F_U$, where $\delta > 0$ and U is a neighborhood of x_o contained in V. Then

$$\| <f|g> - \check{a}[\pi(<f|g>)] \| = \| <f|g> - \check{a}(x) \| = \| <f|g> - <\hat{u}(x)|\hat{v}(x)> \|$$
$$\leqslant \| <f-\hat{u}(x)|g> \| + \| <\hat{u}(x)|g-\hat{v}(x)> \| \leqslant \| f-\hat{u}(x) \| (\| g-\hat{v}(x) \|$$
$$+ \| \hat{v}(x) \|) + \| \hat{u}(x) \| \| g-\hat{v}(x) \| \leqslant \delta(\delta + \| \hat{v}(x) \| + \| \hat{u}(x) \|) < \varepsilon$$

(under suitable choice of δ and U). Moreover, $x = \pi(<f|g>) \in V$.

Thus $<f|g> \in \mathcal{T}_{\varepsilon}(\check{a}) \cap E_V$.

3.11. <u>Definition</u>. The pair (π, ρ) is called the *field of Hilbert modules associated* with the A-Hilbert module H.

Let H (resp. K) be a Hilbert module over a C^*-algebra A (resp. B) with identity. A *Hilbert module isomorphism* of H onto K is a pair (φ,T) of maps

$$A \overset{\varphi}{\to} B \quad , \quad H \overset{T}{\to} K$$

where φ la a C^*-algebra isomorphism of A onto B, and T is such that for all $a \in A$ and $u,v \in H$:

(i) $T(au+v) = \varphi(a)Tu + Tv$ (i.e. T is φ-linear),

(ii) $\varphi(\langle u|v\rangle) = \langle Tu|Tv\rangle$ (i.e., T "preserves" inner products)

(iii) $T(H) = K$ (i.e. T is surjective).

Note that, because of (ii) , $\|u\|^2 = \|\langle u|u\rangle\| = \|\varphi(\langle u|u\rangle)\| = \|\langle Tu|Tu\rangle\| = \|Tu\|^2$ i.e. T is an isometry. In particular, T is injective.

Given a Hilbert module H over A, let (π,ρ) be its associated field of Hilbert modules. Write φ (resp. T) for the map $\check{}$ (resp. $\hat{}$), i.e.

$$A \to \Gamma(\pi) \quad , \quad H \to \Gamma(\rho)$$

$$a \mapsto \check{a} \qquad\qquad u \mapsto \hat{u} \quad .$$

We know that φ is a C^*-algebra isomorphism. Moreover, for any x in X:

(i) $T(au+v)(x) = (au+v)\hat{}(x) = au+v+H_x = (a+m(x))$

$(u+H_x)+(v+H_x) = \check{a}(x)\,\hat{u}(x)+\hat{v}(x) = (\check{a}\hat{u}+\hat{v})(x)$

$= (\varphi(a)\ Tu + Tv)(x)$. Then $T(au+v) = \varphi(a)\ Tu+Tv$.

(ii) $\varphi(<u|v>)(x) = <u|v>^{\vee}(x) = <u|v>+m(x)$

$= <u+H_x|v+H_x> = <\hat{u}|\hat{v}>(x) = <Tu|Tv>(x)$.

Hence $\varphi(<u|v>) = <Tu|Tv>$.

3.12. The Representation Theorem. *Any Hilbert module is isomorphic to the Hilbert module of all continuous sections in the field associated with it.*

Proof. Let $\hat{H} = \{\hat{u}: u \in H\} = T(H)$. This is a $\Gamma(\pi)$-submodule of $\Gamma(\rho)$ (because T is φ-linear) and it is closed because H is complete and T is isometric. Thus, the proof of the theorem reduces to show that $\hat{H}$ is dense in $\Gamma(\rho)$.

Take $\sigma \in \Gamma(\rho)$ and $\varepsilon > 0$ arbitrary. For each $x \in X$ let $u_x \in H$ be such that $\hat{u}_x(x) = \sigma(x)$ and let $V_x = \{y \in X : \|\sigma(y)-\hat{u}_x(y)\| < \varepsilon\}$. Since $\{V_x : x \in X\}$ is an open covering of X, there exist $x_1,\ldots,x_n \in X$ such that $X = V_1 \cup \ldots \cup V_n$ (where $V_i = V_{x_i}$). We can also find $f_1,\ldots,f_n \in C(X)$ with $0 \leqslant f_i \leqslant 1$, $\operatorname{supp} f_i \subseteq V_i$ and $\sum_{i=1}^{n} f_i = 1$; since $f_i\check{1}$ belongs to $\Gamma(\pi) = \check{A}$ we can choose $a_i \in A$ such that $\check{a}_i = f_i \cdot 1$. Define $v_i = a_i u_i (\in H)$, where $u_i = u_{x_i}$, $i = 1,\ldots,n$; then $\hat{v}_i = Tv_i$ $= T(a_i u_i) = \varphi(a_i)(T u_i) = \check{a}_i \hat{u}_i = f_i \hat{u}_i$. Thus, if $v = v_1 + \ldots + v_n (\in H)$ we have $\hat{v} \in \hat{H}$ and $\hat{v} = \hat{v}_1 + \ldots + \hat{v}_n = f_1 \hat{u}_1 + \ldots + f_n \hat{u}_n$. Then, for each $x \in X$:

$$\|\sigma(x) - \hat{v}(x)\| = \|\sum_{i=1}^{n} f_i(x)(\sigma(x) - \hat{u}_i(x))\|$$

$$\leqslant \Sigma f_i(x)\|\sigma(x) - \hat{u}_i(x)\| < (\Sigma f_i(x))\varepsilon = \varepsilon.$$

Thus $\|\sigma - \hat{v}\| < \varepsilon$. This proves that $\hat{H} = T(H)$ is dense in $\Gamma(\sigma)$. But $\hat{H}$ is closed so it has to coincide with $\Gamma(\sigma)$.

Then (φ, T) is a Hilbert module isomorphism of the A-Hilbert module H onto the $\Gamma(\pi)$-Hilbert module $\Gamma(\rho)$. ∎

This theorem provides an approach to the study of Hilbert modules. It is particulary useful when the stalks in the associated field are somewhat simpler or well known. This is the case when A is commutative because then the stalks become ordinay Hilbert spaces :

3.13. <u>Corollary</u>. (The commutative case). *If* H *is a Hilbert module over a commutative* C^**-algebra* A *with identity then its associated field* ρ *is a field of Hilbert spaces whose base space is the* *maximal ideal space of* A, *and* H *is isomorphic to the Hilbert module of all continuos sections of* ρ.

***

BIBLIOGRAPHY

[1] Dauns, J. and Hoffmann, K.H., "Representations of rings by sections", Mem. A.M.S. 83 (1968).

[2] Dixmier, J., "Les C^*-algèbres et leur représentations", Gauthier-Villars, Paris (1964).

[3] —— and Douady, A., "Champs continus d'espaces hilbertiens et de C^*-algèbres", Bull. Soc. Math. France, 91 (1963), 227-284.

[4] Godement, R., "Sur la théorie des représentations unitaires", Ann. of Math. 53 (1951), 68-124.

[5] Kaplansky, I., "Projections in Banach algebras", Ann. of Math. 53 (1951), 235-249.

[6] Kaplansky, I., "Modules over operator algebras" Amer. J. of Math. 75 (1953), 839 - 853.

[7] Takahashi, A., "Duality between Hilbert Modules and Fields of Hilbert Spaces". To appear.

[8] Takahashi, A., "Hilbert Modules over Stonean Algebras". To Appear.

[9] Widom, H., "Embedding in algebras of type I", Duke Math. J. 23 (1956), 309 - 324.

[10] Wright, J.D.M., "A Spectral theorem for normal operators on a Kaplansky-Hilbert module", London Math. Soc. 19 (1969), 258 - 268.

Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá, D.E. COLOMBIA.

(Recibido en enero de 1979)