*Revista Colombiana de Matemáticas*
*Vol. XIII (1979), págs. 63 - 72*

# THE KOLMOGOROV ENTROPY FOR QUANTUM SYSTEMS REVISITED

by

Steven M. Moore

Abstract. In a previous paper [1] we offered a definition of the Kolmogorov entropy for quantum systems. Here we present an analysis of that definition in terms of the quantum theory of measurement to show that it is too restrictive in the sense that it is limited to compatible measurements. We extend that definition to obtain a quantum definition of the Kolmogorov entropy which includes incompatible measurements.

Resumen. En un trabajo anterior [1] ofrecimos una definición de la entropía de Kolmogorov para los sistemas cuánticos. Aqui presentamos un aná-

lisis de esa definición en terminos de la teoría cuántica de medidas para mostrar que es demasiado restringida, en el sentido de que está limitada a medidas compatibles, y extendemos esa definición para obtener una definición cuántica de la entropía de Kolmogorov que incluya medidas incompatibles.

§ 1. Introduction. A definition for quantum systems of the Kolmogorov entropy was first considered in [1]. There are two basic reasons for considering the definition given there as unsatisfactory:

1. We developed the definition indirectly from a representation $\pi_\rho(\mathcal{A})$ of the algebra of observables $\mathcal{A}$ ($\pi_\rho$ was the canonical GNS representation associated with the quantum state $\rho$). We did this in order to be able to use

$$(1) \qquad r(P) = (P\,\Omega_\rho, \Omega_\rho), \quad P \in \pi_\rho(\mathcal{A})$$

as a substitute for a probability measure ($\Omega_\rho$ is the cyclic vector associated with the representation $\pi_\rho$), and hence to appeal to analogies with the classical case. However, it would be more useful to have a direct definition that uses only $\mathcal{A}$ and $\rho$.

2. As we will see, the compatibility condi-

tions used in the definition of [1] imply that we only considered compatible measurements. Since incompatible measurements do exist in quantum mechanics, one should look for a more general definition.

In this paper we generalize our previous definition of the Kolmogorov entropy for quantum systems so that it includes incompatible measurements, In the process we give a more direct definition in terms of the quantum state and the algebra of observables.

As before, we let $\mathcal{A}$ be the $W^*$-algebra which is the algebra of observables of the quantum system [2] and we consider a normal state $\rho$ on $\mathcal{A}$, [3]. $\tau_t$ will denote the continuous $*$-automorphism that implements the time development, i.e. the observable that corresponds to $A \in \mathcal{A}$ at time $t$ is $\tau_t A$ (this was denoted by $\alpha_t$ in [1] ). $\rho$ is assumed to be invariant: $\rho(\tau_t A) = \rho(A)$ for all $t$ and for all $A \in \mathcal{A}$.

To directly associate the entropy with $\mathcal{A}$, $\rho$ and $\tau_t$, we need to define partitions with respect to $\mathcal{A}$. We say that $\alpha = \{P_1, \ldots, P_n\}$ is a *partition* if $\sum P_i = I$, $P_i P_j = \delta_{ij} P_i$, and $P_i \in A$ for each $i$. To make the connection with [1] we note that if $P$ is a projection in $\mathcal{A}$ then $\pi_\rho(P)$ is a projection in $\pi_\rho(\mathcal{A})$, so that if $\alpha$ is a partition in this new sense, then $\pi_\rho(\alpha)$ will be a partition in the old sense *if* we can insure that $\pi_\rho(P)\Omega_\rho \neq 0$, i.e. $\rho(P) \neq 0$. To insure this, we will assume from now on that $\rho$ is faithful.

§ 2. <u>Quantum theory of measurement</u>. An in the classical case, a partition $\alpha$ is a measurement on the quantum system. Von Neumann [4] has shown that the effect of applying $\alpha$ is to change $\rho$ to $\rho_\alpha = \sum \lambda_i \rho_i$, where $\lambda_i = \rho(P_i) > 0$ and $\rho_i(A) = \lambda_i^{-1} \rho(P_i A P_i)$ for each $A \in \mathcal{A}$. Thus, the effect of the measurement $\alpha$ is to turn $\rho$ into a mixture $\rho_\alpha$.

We want to consider only those $\alpha$ for which $\rho_\alpha = \rho$, the reasons being that we want to repeat the measurement $\alpha$ over and over again in order to define the Kolmogorov entropy. If each time we do that, the state changes, then we would get a measurement of a more general randomness and not a measurement of the inherent randomness in the time development itself. Since the classical entropy only measure the randomness present in the time development, we would like to insure that $\rho_\alpha = \rho$.

We would also like the measurement of A and $P_i$ (A self-adjoint in $\mathcal{A}$) to be independent of which one we do first, A or $P_i$. Obviously this is not true in general, since $[P_i, A] \neq 0$ for many $A \in \mathcal{A}$. However, all what is really needed is that $\rho([P_i, A]) = 0$, since this means that as far as the statistics are concerned the order of measuring $A$ and $P_i$ is unimportant because the expectations of $P_i A$ and $A P_i$ are the same.

It is fortunate that these last two concepts are related:

2.1. <u>Proposition</u>. *Let* $\rho$, $\alpha$, $\tau_t$ *be as above and define* $A_\rho = \{A \in \mathcal{A} : \rho([A,B]) = 0 \text{ for all } B \in \mathcal{A}\}$. *Then the following conditions are equivalent:*

a. $\rho_\alpha = \rho_j$

b. $\alpha \subset \mathcal{A}_{\rho_j}$

c. $\tau_t\, \alpha \subset \mathcal{A}_\rho$ *for all* $t$.

<u>Proof</u>. ($a \Rightarrow b$) Let $P_i \in \alpha$, then:

$$\rho(P_i\ A) = \sum_j \rho(P_j\ P_i\ A\ P_j) = \rho(P_i\ A\ P_i)$$

$$\rho(A\ P_i) = \sum_j \rho(P_j\ A\ P_i\ P_j) = \rho(P_i\ A\ P_i)$$

($b \Rightarrow c$) We have:

$$\rho(\tau_t\ P_i \circ A) = \rho(\tau_t(P_i \circ \tau_t^{-1}\ A)) = \rho(P_i \cdot \tau_t^{-1}\ A)$$

$$\rho(A \circ \tau_t\ P_i) = \rho(\tau_t(\tau_t^{-1}\ A \circ P_i)) = \rho(\tau_t^{-1}\ A \cdot P_i)$$

($c \Rightarrow b$) Take $t = 0$

(b $\Rightarrow$ a) We have:

$$\rho(P_i\ A\ P_i) = \rho(P_i([A,P_i] + P_i\ A))$$

$$= \rho(P_i\ [A,\ P_i]) + \rho(P_i\ A)$$

$$= \rho((P_i\ A)P_i - P_i(P_i\ A)) + \rho(P_i\ A)$$

$$= \rho(P_i\ A)$$

Thus:

$$\rho_\alpha(A) = \sum_i \lambda_i \rho_i\ (P_i\ A\ P_i) = \sum_i \rho(P_i\ A\ P_i)$$

$$= \rho((\sum_i P_i)A) = \rho(A)\ .$$

QED.

We say that $\alpha$ is an *admissible partition* if it satisfies one of the above conditions. We note that not all the partitions considered in [1] are admissible. We paid for that generality by having to restrict the Kolmogorov entropy with a compatibility condition. It turns out that the restriction to admissible partitions will allow us to consider incompatible measurements. From now on we only consider admissible partitions.

We note that (b) in last Proposition could equally be written $\alpha'' \in \mathcal{A}_\rho$ where $\alpha''$ is the second commutator of $\alpha$ (the $W^*$-algebra generated by $\alpha$). This suggests an extension: we say a $W^*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is *admissible* if $\mathcal{B} \subset \mathcal{A}_\rho$ .

§ 3. Definition of the Kolmogorov entropy. Our definition of the Kolmogorov entropy $h(\alpha)$ of $\alpha$ is exactly the same as before:

$$
\begin{aligned}
h(\alpha) &= - \sum_i r(\pi_\rho(P_i)) \log_2 r(\pi_\rho(P_i)) \\
&= - \sum_i \rho(P_i) \log_2 \rho(P_i)
\end{aligned}
\tag{2}
$$

The definition of $h(\alpha|\beta)$ given $\alpha = \{P_i\}$, $\beta = \{Q_i\}$ *not* necessarily compatible, will be the basis of our further development. Consider first the original definition for compatible partitions in [1] :

$$
\begin{aligned}
h(\alpha|\beta) &= \sum_j r(\pi_\rho(Q_j)) \sum_i I(r(\pi_\rho(P_i)|\pi_\rho(Q_j)) ) \\
&= \sum_j \rho(Q_j) \sum_i \left[ - \frac{\rho(P_i Q_i)}{\rho(Q_j)} \log \frac{\rho(P_i Q_i)}{\rho(Q_j)} \right] \\
&= \sum_j \rho(Q_j) \sum_i \left[ - \frac{\rho(Q_j P_i Q_j)}{\rho(Q_j)} \log \frac{\rho(Q_j P_i Q_j)}{\rho(Q_j)} \right] \\
&= \sum_j \rho(Q_j) \sum_i I(\rho_j \ (P_i) \ )
\end{aligned}
$$

Thus our definition of $h(\alpha|\beta)$ for any (not necessarily compatible) pair of partitions $\alpha$ and $\beta$

will be

$$(3) \qquad h(\alpha|\beta) = \sum_j \rho(Q_j) \sum_i I(\rho_j(P_i))$$

By mimicking parts of the proof of Theorem 2.2 in [1], one sees that $h(\alpha|\beta)$ still has the properties that we desire of it.

If we can define the complete Kolmogorov entropy in terms of $h(\alpha|\beta)$ (without introducing $\alpha \vee \beta$ ), we will have suceeded in generalizing our previous definition. The clue is in Theorem 3.2 of [1] . To use it, we will need to extend (3) slightly to include admissible subalgebras. If $\mathcal{B}$ is admissible, we define the *conditional entropy* $h(\alpha|\mathcal{B})$ by

$$(4) \qquad h(\alpha|\mathcal{B}) = \inf_{\beta \subset \mathcal{B}} h(\alpha|\beta)$$

Obviously, $\mathcal{B}_1 \subset \mathcal{B}_2$ implies $h(\alpha|\mathcal{B}_1) \geq h(\alpha|\mathcal{B}_2)$ .

Let $T = \tau_1$ and

$$(5) \qquad \mathcal{A}_n(\alpha) = \{T^k\alpha : k = -n, -n+1, \ldots, -1\}''$$

Then we define $h(\alpha, T)$ by

$$(6) \qquad h(\alpha, T) = \lim_{n\to\infty} h(\alpha|\mathcal{A}_n(\alpha))$$

and $h(T)$ by

(7) $$h(T) = \sup h(\alpha, T),$$

the supremum being taken over all partitions with $h(\alpha) < \infty$. As in [1], our basic task is to see that (6) is well-defined.

3.1. Theorem. *The limit in (6) exists.*

Proof. By 2.1, $\alpha \subset \mathcal{A}_\rho$ implies $\mathcal{A}_n(\alpha) \subset \mathcal{A}_\rho$, so $\mathcal{A}_n(\alpha)$ is admissible for each $n$. Since $\mathcal{A}_{n-1}(\alpha) \subset \mathcal{A}_n(\alpha)$, we have $h(\alpha | \mathcal{A}_{n-1}(\alpha)) \geq h(\alpha | \mathcal{A}_n(\alpha))$. Thus $h(\alpha | \mathcal{A}_n(\alpha))$ is a non-increasing sequence of positive numbers. QED.

§ 4. Conclusions. We have seen how our new definition of the Kolmogorov entropy generalizes the old. The advantage of the new definition is that it admits incompatible partitions and refers only to $\mathcal{A}$, $\rho$ and $T$.

Our old questions are still present. What conditions can we put on $\mathcal{A}$ to guarantee that $h(T)<\infty$ ? This is still an unsolved problem. We may note, however, that if anything, we have made it worse, because the new definition of the Kolmogorov entropy given here gives, in general, a larger value for $h(T)$ than the old.

The author wishes to thank *Colciencias* for its aid in financing his trip to the New Orleans Conference on the Mathematical Foundations of Quantum Mechanics, where the results given in this paper began to take shape through discussions related to the measurement problem.

***

## BIBLIOGRAPHY

[1] S.M. Moore, "An Algebraic Study of the Kolmogorov Entropy", Rev. Col. Mat. X (1976), 57 - 68.

[2] —— , "An Algebraic Formulation of Ergodic Problems", Rev. Col. Mat. IX (1975), 189-204.

[3] —— , "Correction to an Algebraic Formulation of Ergodic Problems", Rev. Col. Mat. X (1976), 161.

[4] J. von Neumman, "Grundlagen der Quantenmechanik", Springer, Berlin, (1932).

*Departamento de Física*
*Universidad de los Andes*
*Apartado Aéreo 4976*
*Bogotá, D.E., COLOMBIA.*

(Recibido en noviembre de 1977)